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Cauchy–Gelfand problem for quasilinear conservation law

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A B S T R A C T

We obtain the precise asymptotic ($t \to \infty$) for solution $f(x,t)$ of Cauchy–Gelfand problem for quasilinear conservation law $\frac{\partial f}{\partial t} + \varphi(f)\frac{\partial f}{\partial x} = 0$ with initial data of bounded variation $f(x,0) = f^0(x)$. The main theorem develops results of Liu (1981) [22], Kruzhkov, Petrosjan (1987) [20], Henkin, Shananin (2004) [10], Henkin (2012) [9]. Proofs are based on vanishing viscosity estimates and localized Maxwell type conservation laws. The main application consists in the reconstruction of parameters of initial data responsible for location of inviscid shock waves in the solution $f(x,t)$.

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0. Introduction

We study Cauchy (and inverse Cauchy) problem for equation

$$\frac{\partial f}{\partial t} + \varphi(f)\frac{\partial f}{\partial x} = 0, \quad x \in \mathbb{R},\ t \geqslant 0 \tag{$*$}$$

with initial data $f(x,0) = f^0(x)$. The most natural (not equivalent) definitions for solutions of problem (∗) consists in the existence of solutions $f_\varepsilon \stackrel{\text{def}}{=} f_\varepsilon(x,t)$ for equations

$$\frac{\partial f_\varepsilon}{\partial t} + \varphi(f_\varepsilon)\frac{\partial f_\varepsilon}{\partial x} = \varepsilon\frac{\partial^2 f_\varepsilon}{\partial x^2}, \quad \varepsilon > 0,\ x \in \mathbb{R}, \tag{1a}$$

$$\frac{\partial f_\varepsilon}{\partial t} + \varphi(f_\varepsilon)\frac{f_\varepsilon(x,t) - f_\varepsilon(x-\varepsilon,t)}{\varepsilon} = 0, \quad x \in \mathbb{R}, \tag{1b}$$

with property

$$f_\varepsilon(x,0) = f^0(x), \quad \varepsilon \geqslant 0, \tag{2}$$

such that $f_\varepsilon(x,t) \to f_0(x,t)$, when $\varepsilon \to +0$.

Eq. (1a) with linear $f_\varepsilon \mapsto \varphi(f_\varepsilon)$ was introduced at first by Riemann [26] (for $\varepsilon = +0$) and later by Bateman [1], Burgers [3] and Hopf [17] (for $\varepsilon > 0$) as the simplest approximation to the equations of fluid dynamics.

Eq. (1a) for general $\varphi(f_\varepsilon)$ was introduced later in a very different models: displacements of oil by water, consolidation of wet soil, the road traffic, etc. Eq. (1b) was introduced by, Henkin, Polterovich [12], for description of a Schumpeterian evolution of industry. In physical applications of (1a) the main interest has the inviscid case, when $\varepsilon = +0$, but the application of (1a) in the transport flow theory and of (1b) in Schumpeterian dynamics the main interest presents the viscid case, when $\varepsilon > 0$.

Solutions of Eq. (1a) with $\varepsilon = +0$ are called usually viscosity solutions and were introduced and deeply studied by Hopf [16], Gelfand [7], Lax [21], Oleinik [24], Kruzhkov [19], etc. Detailed conditions for uniqueness and existence of such solutions are given, for example, in Crandall, Evans, Lions [4].

It is important to remark that solutions of (1b) with $\varepsilon = +0$ are not the same as solutions of (1a) with $\varepsilon = +0$, in spite that for $\varepsilon = 0$ the both Eqs. (1a), (1b) look identical. The reason for this difference consists in the fact that Eq. (1b) is a semi-discrete approximation of the nonconservative equation

$$\frac{\partial f_\varepsilon}{\partial t} + \varphi(f_\varepsilon)\frac{\partial f_\varepsilon}{\partial x} = \frac{\varepsilon}{2}\varphi(f_\varepsilon)\frac{\partial^2 f}{\partial x^2}.$$

Assumption 1. Let $\alpha^- < \alpha^+$, $f^0(x)$ be real-valued function of bounded variation on $\mathbb{R}$ such that $f^0(x) = \alpha^\pm$, if $\pm x \geqslant \pm x^\pm$, $x^- < x^+$. Let $\varphi(f)$ be a positive, continuous differentiable function of real variable f such that $\varphi'(f)$ has only isolated zeros.