Bull. Sci. math. 138 (2014) 805–845

Contents lists available at ScienceDirect

Bulletin des Sciences Mathématiques

www.elsevier.com/locate/bulsci

The monotonicity of the ratio of hyperelliptic integrals ☆

Na Wang, Dongmei Xiao *, Jiang Yu

Department of Mathematics, Shanghai Jiao Tong University, Shanghai 200240, China

A R T I C L E   I N F O

Article history:
Received 1 November 2013
Available online 19 February 2014

MSC:
primary 34C08, 34C07
secondary 14K20

Keywords:
Abelian integrals
Monotonicity
Hyperelliptic curves

A B S T R A C T

In this paper, we study the monotonicity of the ratio of two Abelian integrals

$$I_0(h) = \oint_{\gamma(h)} y\,dx, \quad \text{and} \quad I_1(h) = \oint_{\gamma(h)} xy\,dx$$

in an interval Σ, where $\gamma(h)$ is a compact component of hyperelliptic curves with genus 2 as $h \in \Sigma$. We first give the topological classification of these hyperelliptic curves. Then we show what kinds of compact components in the classification ensure that the ratio of the two Abelian integrals is monotone.

1. Introduction

The problem of finding the isolated zeros of Abelian integral

☆ Research was supported by the NNSF of China (Nos. 10925102 & 11371248), the RFDP of Higher Education of China grant (No. 20130073110074) and NSF of JiangSu (BK 20131285).

* Corresponding author.
E-mail address: xiaodm@sjtu.edu.cn (D. Xiao).

http://dx.doi.org/10.1016/j.bulsci.2014.02.001

$$I(h) = \int\limits_{\gamma(h)} f(x,y)\,dx + g(x,y)\,dy, \quad h \in \Sigma$$

is called the infinitesimal Hilbert problem (or weakened Hilbert 16th problem) if $f(x,y)$, $g(x,y)$ are polynomials of degree m, $H(x,y)$ is a polynomial of degree $n+1$, and $\gamma(h)$ is a compact component of the level set $H(x,y) = h$, where $\Sigma \subset R$ is a maximal open interval such that $\gamma(h)$ is the compact component (see [1]).

It is well known that the curve $\gamma(h)$ is called the hyperelliptic curve if the Hamiltonian function $H(x,y)$ has the form $H(x,y) = y^2 + P_{n+1}(x)$ and $g = [\frac{n}{2}] \geqslant 2$, where $P_{n+1}(x)$ is a polynomial of degree $n+1$ and g is called the genus of the hyperelliptic curve. When $g < 2$, the curve $\gamma(h)$ is called the elliptic curve. There have been many fruitful works on the accurate estimation of number of isolated zeros of $I(h)$ if the curve $\gamma(h)$ is elliptic curve with genus one, such as G.S. Petrov's work in [10] for $n = 2$ and the successive works by Dumortier and Li in [2–5] for $n = 3$. However, to our knowledge, there have not been much studies on $I(h)$ if the curve $\gamma(h)$ is the hyperelliptic curve. Clearly, Hamiltonian function $H(x,y)$, whose graph is the hyperelliptic curve with genus two, has two forms: one is $H(x,y) = y^2 + P_5(x)$ and the other is $H(x,y) = y^2 + P_6(x)$, i.e. $n = 4$ and $n = 5$. Gavrilov and Iliev in [6] first studied the hyperelliptic curve with $H(x,y) = y^2 + P_5(x)$ and obtained that the 2-dimensional real vector space consisting of two complete Abelian integrals of the first kind, $\oint_{\gamma(h)} \frac{1}{y}\,dx$ and $\oint_{\gamma(h)} \frac{x}{y}\,dx$, is Chebyshev for exceptional families of the compact component $\gamma(h)$. Later, Liu and Xiao in [9] studied if the 2-dimensional real vector space consisting of two Abelian integrals, $\oint_{\gamma(h)} y\,dx$ and $\oint_{\gamma(h)} xy\,dx$, is Chebyshev for all compact components $\gamma(h)$ of $H(x,y) = y^2 + P_5(x)$. They gave a useful criterion for the monotonicity of the ratio of two Abelian integrals, $\oint_{\Gamma(h)} y\,dx$ and $\oint_{\Gamma(h)} xy\,dx$, where $\Gamma(h)$ is the compact component of $y^2 + \Psi(x) = h$, $\Psi(x)$ is analytic with a local minimum at the center of a Hamiltonian system. As an application of their criterion, they obtained the sufficient and necessary conditions for Chebyshev property of the vector space of two Abelian integrals, $\oint_{\gamma(h)} y\,dx$ and $\oint_{\gamma(h)} xy\,dx$, where $\gamma(h)$ is the compact component of $y^2 + P_5(x) = h$ which surrounds only one center. It is clear that if the ratio of two Abelian integrals, $\oint_{\gamma(h)} y\,dx$ and $\oint_{\gamma(h)} xy\,dx$, is monotone, then the Abelian integral $I(h)$ has at most one isolated zero when $f(x,y) = (\beta_0 + \beta_1 x)y$ and $g(x,y) = 0$.

The aim of this paper is to study the monotonicity of the ratio of two Abelian integrals, $\oint_{\gamma(h)} y\,dx$ and $\oint_{\gamma(h)} xy\,dx$, where $\gamma(h)$ is the compact component of $y^2 + P_6(x) = h$ which has only real critical points.

We consider the corresponding Hamiltonian systems as follows

$$(X_H^-)\colon \quad \begin{cases} \dot{x} = -2y, \\ \dot{y} = p(x), \end{cases} \quad \text{or} \quad (X_H^+)\colon \quad \begin{cases} \dot{x} = 2y, \\ \dot{y} = p(x), \end{cases}$$

where $p(x) = \frac{dP_6(x)}{dx}$, which is a real polynomial of degree five with five real critical points (counted with multiplicities).