Bull. Sci. math. 138 (2014) 860–869

Contents lists available at ScienceDirect

Bulletin des Sciences Mathématiques

www.elsevier.com/locate/bulsci

Measure preserving homeomorphism of the Hilbert cube embedding the pseudo-boundary into the pseudo-interior ☆

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ARTICLE INFO

Article history:
Received 6 November 2013
Available online 11 March 2014

MSC:
28A65
54C05

Keywords:
Hilbert cube
Measure-preserving mapping
Pseudo-interior
Pseudo-boundary

ABSTRACT

We prove that the pseudo-boundary $B(Q)$ of the Hilbert cube Q can be embedded into the pseudo-interior s of Q by a homeomorphism of Q that preserves the product Lebesgue measure.

1. Introduction

G. Cantor showed that for two countable and dense subsets A, B of the real line $\mathbb{R}$ there exists a homeomorphism h of the real line transforming A onto B. This property of $\mathbb{R}$ and any other topological space is called *countable dense homogeneity*. Countable

☆ The work has been partially financed by NCN means granted by decision DEC-2011/01/B/ST1/01439.

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http://dx.doi.org/10.1016/j.bulsci.2014.03.001
0007-4497/

dense homogeneity of all Euclidean spaces $\mathbb{R}^n$, $n \in \mathbb{N}$, was showed by L.E.J. Brouwer [4]. Later Brouwer's result was generalized to manifolds by R.B. Bennett [2]. M. Fort [7] proved that also the Hilbert cube Q is countable dense homogeneous. P. Franklin [8] showed that a homeomorphism in Cantor's result can be an analytic function. (A bit weaker result with a diffeomorphism in the conclusion was independently obtained by Z. Zalcwasser [15].) M. Morayne [11] showed that a homeomorphism in Brouwer's result for $\mathbb{R}^n$, $n > 1$, can be an analytic diffeomorphism preserving the n-dimensional Lebesgue measure and for the n's even it can be an analytic diffeomorphism of the canonical complex space $\mathbb{C}^{\frac{n}{2}}$. This last result was later independently proved by J.-P. Rosay and W. Rudin [14]. M. Pietroń [13] showed that also a homeomorphism in Fort's result can preserve the product Lebesgue measure on Q.

Basic tools for investigating the topology of infinite dimensional spaces, including the Hilbert cube Q, are so called $\mathcal{Z}$-sets and their countable unions $\mathcal{Z}_\sigma$-sets. (For a metric space (X,d) we say that $A \subseteq X$ is a *$\mathcal{Z}$-set* if for an arbitrary $\varepsilon > 0$ and an arbitrary continuous mapping $f : Q \to X$ there exists another continuous mapping $g : Q \to X \setminus A$ such that $\sup\{d(f(a), g(a)):\ a \in Q\} < \varepsilon$.) Two subsets A, B of a topological space X are *topologically equivalent* (*ambiently homeomorphic*) in X if there exists a homeomorphism $f : X \to X$ of X onto X such that $f[A] = B$. Topological equivalence of particular $\mathcal{Z}_\sigma$-sets in the Hilbert cube is often an important question. Always an important example of a $\mathcal{Z}_\sigma$-set is a countable dense subset of X. Investigating the topological equivalence of all such sets in a space X, i.e. the countable dense homogeneity of X, was shortly described for various X's in the previous paragraphs (see also [5,9]). For the Hilbert cube another very important $\mathcal{Z}_\sigma$-set is its pseudo-boundary $B(Q)$. It is known that any $\mathcal{Z}_\sigma$-set can be embedded into the pseudo-interior s of the Hilbert cube by a homeomorphism of the whole Hilbert cube that is arbitrarily close to the identity. It is especially useful for the pseudo-boundary of the Hilbert cube (which is a $\mathcal{Z}_\sigma$-set). We will prove that such a homeomorphism transforming $B(Q)$ into s can additionally preserve the product Lebesgue measure. How far this result can be generalized replacing the pseudo-boundary by other $\mathcal{Z}_\sigma$-sets we do not know.

2. Definitions, notation, auxiliary results

The main space considered in this article is the *Hilbert cube* $Q = [0,1]^{\mathbb{N}}$ with the natural product topology.

Let (X,d) be any metric space where the metric d is bounded.

For $A, B \subseteq X$ let

$$\hat{d}(A,B) = \inf\{d(a,b):\ a \in A, b \in B\}.$$

With some abuse of notation for $A \subseteq X$ and $x \in X$ we shall write:

$$\hat{d}(x,A) = \hat{d}(\{x\},A).$$