Bull. Sci. math. 138 (2014) 870–886

Contents lists available at ScienceDirect

Bulletin des Sciences Mathématiques

www.elsevier.com/locate/bulsci

# Quadratic overgroups for diamond Lie groups ☆

Lobna Abdelmoula *, Yasmine Bouaziz

*Department of Mathematics, Faculty of Sciences at Sfax, Sfax University, Soukra Street, 3000, Sfax, Tunisia*

ARTICLE INFO

*Article history:*
Received 21 October 2013
Available online 12 April 2014

*MSC:*
primary 22E27
secondary 32G05

*Keywords:*
Diamond Lie group
Quadratic overgroup
Moment set

ABSTRACT

Let $G$ be a connected and simply connected solvable Lie group. The moment map for $\pi$ in $\widehat{G}$, unitary dual of $G$, sends smooth vectors in the representation space of $\pi$ to $\mathfrak{g}^*$, dual space of $\mathfrak{g}$. The closure of the image of the moment map for $\pi$ is called its *moment set*, denoted by $I_\pi$. Generally, the moment set $I_\pi$, $\pi \in \widehat{G}$ does not characterize $\pi$, even for generic representations. However, we say that $\widehat{G}$ is *moment separable* when the moment sets differ for any pair of distinct irreducible unitary representations. In the case of an exponential solvable Lie group $G$, D. Arnal and M. Selmi exhibited an accurate construction of an overgroup $G^+$, containing $G$ as a subgroup and an injective map $\Phi$ from $\widehat{G}$ into $\widehat{G^+}$ in such a manner that $\Phi(\widehat{G})$ is moment separable and $I_{\Phi(\pi)}$ characterizes $\pi$, $\pi \in \widehat{G}$. In this work, we provide the existence of a quadratic overgroup for the diamond Lie group, which is the semi-direct product of $\mathbb{R}^n$ with $(2n+1)$-dimensional Heisenberg group for some $n \geqslant 1$.



☆ This work was completed with the support of D.G.R.S.R.T. through the Research Laboratories: LR 11 ES 35 and LR 11 ES 52.

* Corresponding author.
*E-mail addresses:* Lobnaabdelmoula@yahoo.fr (L. Abdelmoula), bouaziz.yasmine08@yahoo.fr (Y. Bouaziz).

http://dx.doi.org/10.1016/j.bulsci.2014.04.001
0007-4497/

## 1. Introduction

Let $G$ be a real Lie group with Lie algebra $\mathfrak{g}$ and $\pi$ be a unitary representation of $G$ in some Hilbert space $\mathcal{H}_\pi$. The moment map for $\pi$ is defined as

$$\Psi_\pi : \mathcal{H}_\pi^\infty \setminus \{0\} \to \mathfrak{g}^*, \quad \Psi_\pi(\xi)(X) = \frac{1}{i}\frac{\langle d\pi(X)\xi, \xi\rangle}{\langle \xi, \xi\rangle}, \quad X \in \mathfrak{g},\ \xi \in \mathcal{H}_\pi^\infty \setminus \{0\}, \tag{1.1}$$

where $d\pi$ denotes the derived representation of $\mathfrak{g}$ in the space $\mathcal{H}_\pi^\infty$ of smooth vectors. When $\pi$ is finite dimensional, this notation which is due to N. Wildberger, reduces to that of the usual moment map of Hamiltonian action of $G$ via $\pi$ on the projective space $P(\mathcal{H}_\pi)$ [9]. The moment set for $\pi$ is defined as

$$I_\pi = \overline{\{\Psi_\pi(\xi),\ \xi \in \mathcal{H}_\pi^\infty \setminus \{0\}\}},$$

the closure of the image of $\Psi$ in $\mathfrak{g}^*$. We denote $\widehat{G}$ the set of irreducible unitary representations of $G$ up to unitary equivalence and consider the moment sets $I_\pi$ for $\pi \in \widehat{G}$. One says that $\widehat{G}$ is *moment separable* if $I_\pi \neq I_\rho$ for all $\pi$ and $\rho \in \widehat{G}$ with $\pi \not\simeq \rho$. A fundamental result of Wildberger related the moment set $I_\pi$ for $\pi \in \widehat{G}$ when $G$ is connected and simply connected nilpotent Lie group to the coadjoint orbit $\mathcal{O}_\pi \subset \mathfrak{g}^*$ associated to $\pi$ via Kirillov method [9]. Namely,

$$I_\pi = \overline{\mathrm{Conv}}(\mathcal{O}_\pi), \tag{1.2}$$

the closure of the convex hull of $\mathcal{O}_\pi$ in $\mathfrak{g}^*$. This result has been generalized to encompass connected and solvable Lie group by D. Arnal and J. Ludwig [3]. In view of Eq. (1.2), $\widehat{G}$ is moment separable if and only if

$$\overline{\mathrm{Conv}}(\mathcal{O}_\pi) = \overline{\mathrm{Conv}}(\mathcal{O}_\rho) \quad \Rightarrow \quad \mathcal{O}_\pi = \mathcal{O}_\rho,$$

for all coadjoint orbits $\mathcal{O}_\pi, \mathcal{O}_\rho \subset \mathfrak{g}^*$.

A counterexample given by Wildberger shows that $\widehat{G}$ need not be moment separable even for nilpotent connected simply connected Lie group. For thus, A. Baklouti, J. Ludwig and M. Selmi in [7] extended the moment map to the dual of the complex universal enveloping algebra $\mathcal{U}(\mathfrak{g}^{\mathbb{C}})$ of the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$ and in [4], it is shown that the generalized moment set characterizes the unitary irreducible representation of any connected and simply connected exponential Lie group.

Concerning the general nonexponential setting, the problem of separation of irreducible unitary representations of a connected and simply connected Lie group $G$ is advocated by L. Abdelmoula and all in [2] where they have shown that the generalized moment sets always separate the elements of the unitary dual $\widehat{G}$.

However, unlike the usual moment map $\Psi_\pi$, $\pi \in \widehat{G}$, this generalized moment map has no natural geometrical interpretation. It is thus natural to seek a new way to separate