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On a kind of symmetric weakly non-linear ordinary differential systems

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ABSTRACT

For a class of weakly non-linear ordinary differential equations, the existence of a unique symmetric solution is established and its stability is studied. The symmetry of a solution is understood in the sense of a certain linear functional equality which includes, in particular, the cases of periodic, anti-periodic, even, and odd solutions. Efficient stability conditions in terms of logarithmic norms and spectral stability conditions are obtained. The theory is illustrated by examples.

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1. Introduction

The present work is concerned with the qualitative analysis of a system with a small parameter under a certain symmetry assumption on the non-linearity. More precisely, we deal with the system of n weakly non-linear ordinary differential equations

$$x'(t) = \varepsilon f(x(t), t), \qquad t \in J, \tag{1}$$

where $J \subset \mathbb{R}$ is an open interval, $\varepsilon \in \mathbb{R}$ is a small parameter, and the function $f : \mathbb{R}^n \times J \to \mathbb{R}^n$ is continuous and symmetric in the sense that there exist a regular matrix $A : \mathbb{R}^n \to \mathbb{R}^n$ and a function $\psi \in C^1(J, J)$ such that the relation

$$Af(z, t) = \psi'(t) f(Az, \psi(t)) \tag{2}$$

holds for all $(z, t) \in \mathbb{R}^n \times J$. We also suppose in the sequel that $f(\cdot, t)$ is C^2-smooth for every $t \in J$.

Equation (1) is considered in an arbitrary but fixed open bounded subset of $\mathbb{R}^n$ for the space variable (we thus do not study bifurcations from infinity of (1) for ε small).

The main object of the present study is the class of solutions of system (1) that are symmetric in the following sense.

Definition 1.1. By a *symmetric solution* x of equation (1), we understand a function $x \in C^1(J, \mathbb{R}^n)$ satisfying (1) on J and possessing the property

$$x(\psi(t)) = Ax(t), \qquad t \in J. \tag{3}$$

The symmetry condition (2) on the non-linearity, which is assumed to be satisfied throughout the paper, arises in a natural way due to our interest in property (3). For example, in the T-periodic case (i.e., when A is the unit matrix, $J = \mathbb{R}$, and ψ is the additive shift by T), condition (2) means that $f(z, \cdot)$ is T-periodic for any z. For $A = I$, $J = \mathbb{R}$, and $\psi(s) = -s$, $s \in J$, the property mentioned leads one to the important class of time-reversible systems (see, e.g., [1,2]).

The structure of property (3) is rather rich and, apart from periodicity, it obviously covers other widely studied types of solutions, e.g., the even and odd solutions ($A = \pm I$, $\psi(t) \equiv -t$, $J = -J$). Meaningful examples can be indicated even for the scalar functions, in which case (3) has the form

$$x(t) = ax(\psi(t)), \qquad t \in J, \tag{4}$$

with some $a \in \mathbb{R} \setminus \{0\}$. One can easily check that property (4) holds, for instance, with the following choices of x, ψ, and a and a suitable J:

$$x(t) = (t + \tau)^m, \qquad \psi(t) = a^{-1/m} t + \beta, \qquad a \in (0, 1) \cap (1, \infty)$$