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# The period function and the Harmonic Balance Method

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ABSTRACT

In this paper we consider several families of potential non-isochronous systems and study their associated period functions. Firstly, we prove some properties of these functions, like their local behavior near the critical point or infinity, or their global monotonicity. Secondly, we show that these properties are also present when we approach to the same questions using the Harmonic Balance Method.



## 1. Introduction and main results

Given a planar differential system having a continuum of periodic orbits, its period function is defined as the function that associates with each periodic orbit its period. To determine the global behavior of this period function is an interesting problem in the qualitative theory of differential equations either as a theoretical question or due to its appearance in many situations. For instance, the period function is present in

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mathematical models in physics or ecology, see [13,32,35] and the references therein; in the study of some bifurcations [9, pp. 369–370]; or to know the number of solutions of some associated boundary value problems, see [6,7].

In particular, there are several works giving criteria for determining the monotonicity of the period function associated with some systems, see [6,15,18,33,37] and the references therein. Results about non-monotonous period functions have also recently appeared, see for instance [17,20,24].

The so-called $N$-th order Harmonic Balance Method (HBM) consists in approximating the periodic solutions of a non-linear differential equation by using truncated Fourier series of order $N$. It is mainly applied with practical purposes, although in many cases there is no theoretical justification. In most of the applications this method is used to approach isolated periodic solutions, see for instance [16,22,27–30]. Since the HBM also provides an approximation of the angular frequency of the searched periodic solution, it can be used to get its period as well.

Hence, applying the HBM to systems of differential equations having a continuum of periodic orbits we can obtain approximations of the corresponding period functions. The main goal of this paper is to illustrate this last assertion through the study of several concrete planar systems. This approach is also used for instance in [2,3,31]. A main difference between these works and our paper is that we also carry out a detailed analytic study of the involved period functions.

More specifically, in this work we will consider several families of planar potential systems, $\ddot{x} = f(x)$, having continua of periodic orbits. We will study analytically their corresponding period functions and we will see that the approximations of the period functions obtained using the $N$-th order HBM, for $N = 1$, keep the essential properties of the actual period functions: local behavior near the critical point and infinity, monotonicity, oscillations, etc. For the case of the Duffing oscillator we also consider $N = 2$ and 3. In particular, the method that we introduce using resultants gives an analytic way to deal with the third order HBM, answering question (iv) in [31, p. 180].

First, we focus on the following two families of potential differential systems:

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = x + x^{2m-1}, \quad m \in \mathbb{N} \text{ and } m \geq 2, \end{cases} \tag{1}$$

and

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -\dfrac{x}{(x^2 + k^2)^m}, \quad k \in \mathbb{R} \setminus \{0\},\ m \in [1, \infty). \end{cases} \tag{2}$$

Each system of these families has a continuum of periodic orbits around the origin. Thus, we can talk about its period function $T$ which associates with each periodic orbit passing through $(x, y) = (A, 0)$ its period $T(A)$. In addition, we will denote by $T_N(A)$ the approximation to $T(A)$ by using $N$-th order HBM; see Section 2.2 for the precise definition of $T_N(A)$.