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Jeu de taquin and diamond cone for Lie (super)algebras

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A B S T R A C T

In this paper, we recall combinatorial basis for shape and reduced shape algebras of the Lie algebras $\mathfrak{gl}(n)$, $\mathfrak{sp}(2n)$ and $\mathfrak{so}(2n+1)$. They are given by semistandard and quasistandard tableaux. Then we generalize these constructions to the case of the Lie superalgebra $\mathfrak{spo}(2n, 2m+1)$. The main tool is an extension of Schützenberger's jeu de taquin to these algebras.

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1. Introduction

The notion of tableaux was introduced by A. Young in 1900. They were used by G. Frobenius to study the representation of the symmetric groups in 1903. They became a fundamental object to study the representation of general linear groups. For $n \in \mathbb{Z}^+$, and a partition λ of n, semistandard tableaux with shape λ give an explicit basis of the simple $\mathfrak{gl}(n)$-modules. The notion of semistandard tableaux was thus generalized to the case of the classical simple Lie algebras $\mathfrak{sl}(n)$ (see [12,4]), $\mathfrak{sp}(2n)$ (see [11,15,19]), $\mathfrak{so}(2n)$ and $\mathfrak{so}(2n+1)$ (see [15]). As for $\mathfrak{gl}(n)$, for a fixed shape λ, a basis of each simple module associated to λ is indexed by semistandard Young tableaux with corresponding type.

The sum of all simple modules of a semisimple Lie algebra has a structure of commutative associative algebra, called the shape algebra. This algebra has many realizations: geometric, algebraic and combinatorics. The Weyl construction realizes each simple module as a submodule of the tensor algebra of the natural module using the Schur functors (for details see [12]).

This notion was extended in the case of the Lie superalgebras $\mathfrak{sl}(n,m)$ and $\mathfrak{spo}(2n,2m+1)$ (see for instance [6,10]). For $\mathfrak{sl}(n,m)$, they give a basis for the sum of all simple tensor modules.

Let $\mathfrak{g}$ be a semisimple Lie algebra. Denote by $\mathfrak{n}$ the nilpotent factor in the splitting of $\mathfrak{g}$. Each simple module is generated by its (normalized) highest weight vector v_λ. The reduced shape algebra of $\mathfrak{g}$ is the quotient of the shape algebra by the ideal generated by the $v_\lambda - 1$. This algebra is an indecomposable $\mathfrak{n}$-module, which is the union of natural monogenic $\mathfrak{n}$-modules. For $\mathfrak{n}$, the reduced shape algebra appears as the main tool to describe the nilpotent $\mathfrak{n}$-modules.

Suppose first $\mathfrak{g} = \mathfrak{sl}(n)$. Each highest weight vector corresponds to a trivial tableau ($t_{i,j} = i$ for any i, j). A good basis for the reduced shape algebra (the diamond cone) is the collection of all quasistandard Young tableaux (see [4]). Essentially a quasistandard tableau is a semistandard one from which it is impossible to 'extract' any trivial subtableau by using Schützenberger's jeu de taquin.

The good generalization of this jeu de taquin, to the cases $\mathfrak{g} = \mathfrak{sp}(2n)$ and $\mathfrak{g} = \mathfrak{so}(2n+1)$ was given by Lecouvey [15,16]. Using this generalization, it is possible to build the diamond cone for these Lie algebras. Symplectic and orthogonal quasistandard tableaux can be thus defined. They give diamond cones for these Lie algebras. Similarly a super jeu de taquin was introduced in [2] and used to define quasistandard $\mathfrak{sl}(m,n)$-tableaux. The symplectic jeu de taquin was also considered to study the reduced shape algebra for the Lie superalgebra $\mathfrak{spo}(2n,1)$ [3].

The goal of the present paper is multiple: first present the reduced shape algebra for a general semisimple Lie algebra $\mathfrak{g}$, and its relation with nilpotent $\mathfrak{n}$-module, then describe semistandard and quasistandard tableaux for $\mathfrak{gl}(n)$, $\mathfrak{sp}(2n)$, and $\mathfrak{so}(2n+1)$. Finally, study the case of the Lie superalgebra $\mathfrak{spo}(2n,2m+1)$ (defined in [13]).