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Hitchin pairs on an integral curve

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Abstract

Let Y be an integral projective scheme of dimension 1 over a field k (of arbitrary characteristic). We construct the moduli spaces of Hitchin pairs on Y. We define a notion of strong semistability for Hitchin pairs on Y. Assume that the base field is infinite and the normalization of Y is smooth. Then we show that finite direct sums of strongly semistable Hitchin pairs on Y form a neutral Tannakian category. We define the holonomy group scheme $\mathcal{G}_Y^H$ of Y to be the Tannakian group scheme for this category. For a strongly semistable G-Hitchin pair, we construct a monodromy group scheme.

We study the relation between Higgs bundles on Y and the representations of the (topological) fundamental group of Y in the complex general linear group.

MSC: primary 14H60; secondary 14F05

Keywords: Hitchin pairs; Curve; Tannakian category

1. Introduction

Hitchin pairs (or Higgs bundles) on smooth varieties have been studied extensively by algebraic geometers as well as differential geometers in last few decades (e.g. [13,14]). The notion extends naturally to singular curves (Definition 2.1). In this paper we study some aspects of the theory of Hitchin pairs and Hitchin G-bundles (G a reductive group) on an integral projective curve Y defined over any field k.

Following [4], we define a notion of strong semistability for Hitchin pairs (and G-Hitchin pairs) on Y (Definitions 3.4, 6.2). It coincides with the usual strong semistability in case Y is

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smooth. In case Y is singular, it is different from semistability in characteristic 0 (there exist semistable Hitchin pairs which are not strongly semistable). In characteristic $p > 0$, it turns out that strong semistability is the same as Frobenius strong semistability (Corollary 3.8).

We construct a category $\mathcal{C}_Y^{\mathcal{H}}$ with objects finite direct sums of strongly semistable Hitchin bundles on Y. Assume that the normalization X of Y is smooth and Y has a k-rational point. Then we prove that $\mathcal{C}_Y^{\mathcal{H}}$ is a neutral Tannakian category over k (see Section 4 for details). Let $\mathcal{G}_Y^{\mathcal{H}}$ denote the group scheme defined over k dual to the neutral Tannakian category $\mathcal{C}_Y^{\mathcal{H}}$. This group scheme is an invariant of Y. We call it the "Hitchin holonomy group scheme" of Y. There is also a tautological $\mathcal{G}_Y^{\mathcal{H}}$-bundle $E_{\mathcal{G}_Y}$, called the Hitchin holonomy bundle (Section 5). These results are generalized to Hitchin G-bundles (Section 6).

In Section 7, we specialize to $k = \mathbb{C}$ and Y an irreducible complex curve with m ordinary nodes as singularities. We investigate the relation between strongly semistable Hitchin bundles of rank n, degree 0 on Y and Hitchin bundles associated to representations of the (topological) fundamental group of Y in $\mathrm{GL}(n)$.

In final section, we construct the moduli spaces of Hitchin pairs on Y.

2. Hitchin pairs

Let Y be an integral projective curve over k and L be a locally free sheaf of $\mathcal{O}_Y$-modules on Y.

Definition 2.1. An (L-twisted) Hitchin pair $(\mathcal{E}, \phi)$ is a coherent sheaf $\mathcal{E}$ of $\mathcal{O}_Y$-module on Y together with a morphism $\phi : \mathcal{E} \to \mathcal{E} \otimes L$.

Definition 2.2. A Hitchin bundle (E, ϕ) is a vector bundle E over Y together with a morphism $\phi : E \to E \otimes L$.

Note. When L is the canonical bundle, a Hitchin pair is called a *Higgs pair* and a Hitchin bundle is called a *Higgs bundle*.

Definition 2.3. A *Hitchin pair* $(\mathcal{E}, \phi)$ is called semistable (resp. stable) if for any proper subsheaf $\mathcal{F}$ of $\mathcal{E}$ which is ϕ invariant in the sense that $\phi(\mathcal{F}) \subset \mathcal{F} \otimes L$, we have

$$\mu(\mathcal{F}) = \frac{degree(\mathcal{F})}{rank(\mathcal{F})} \leqslant \frac{degree(\mathcal{E})}{rank(\mathcal{E})} = \mu(\mathcal{E})$$

(resp. $< \mu(E)$), where $\mu(E)$ is the slope of $\mathcal{E}$.

Note. It is clear that if $\mathcal{E}$ is a semistable (resp. stable) coherent sheaf then $(\mathcal{E}, \phi)$ is a semistable (resp. stable) Hitchin pair for all ϕ.

Definition 2.4. Let (E, ϕ) and (F, ψ) be Hitchin bundles. Their tensor product $(E \otimes F, \phi \otimes \psi)$ is defined as follows: $E \otimes F$ is the tensor product of bundles and for $x \in E$, $y \in F$ we have,

$$\phi \otimes \psi : E \otimes F \to (E \otimes F) \otimes L,$$
$$x \otimes y \mapsto \phi(x) \otimes y + x \otimes \psi(y).$$

Then for integers $n > 1$,