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A note on the growth of Betti numbers and ranks of 3-manifold groups

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Abstract

Let N be an irreducible, compact 3-manifold with empty or toroidal boundary which is not a closed graph manifold. We show that it follows from the work of Agol, Kahn–Markovic and Przytycki–Wise that $\pi_1(N)$ admits a cofinal filtration with 'fast' growth of Betti numbers as well as a cofinal filtration of $\pi_1(N)$ with 'slow' growth of ranks.

1. Introduction

A *filtration of a group* π is a sequence $\{\pi_i\}_{i\in\mathbb{N}}$ of finite index subgroups of π such that $\pi_{i+1} \subset \pi_i$ for every i. We say that a filtration is *cofinal* if $\bigcap_{i\in\mathbb{N}} \pi_i$ is trivial, we call it *normal* if $\pi_i \lhd \pi$ for every i, and we say it is *almost normal* if there exists a k such that $\pi_i \lhd \pi_k$ for every $i \geqslant k$. A group which admits a cofinal normal filtration is called *residually finite*.

Given a filtration $\{\pi_i\}_{i\in\mathbb{N}}$ of a group π it is of interest to study how the following measures of 'complexity' grow:

(1) the first Betti number $b_1(\pi_i) = \dim H_1(\pi_i; \mathbb{Q})$,
(2) the $\mathbb{F}_p$-Betti numbers $b_1(\pi_i, \mathbb{F}_p) = \dim H_1(\pi_i; \mathbb{F}_p)$,
(3) the rank $d(\pi_i)$, i.e. the minimal size of a generating set,
(4) the order of Tor $H_1(\pi_i; \mathbb{Z})$.

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Such growth functions have been studied for 3-manifold groups by many authors over the years. We refer to [4–9,13,15–17,20,19,12,25–30] for a sample of results in this direction. It is clear that given any group π we have $d(\pi) \geqslant b_1(\pi)$, i.e. given a filtration the ranks grow at least as fast as the Betti numbers.

Now let N be a 3-manifold. Throughout this paper we will use the following convention: a 3-manifold will always be assumed to be connected, compact, orientable and irreducible with empty or toroidal boundary. By [10] the group $\pi_1(N)$ is residually finite. In this paper we are interested in how fast Betti numbers can grow in a cofinal filtration of $\pi_1(N)$ and how slowly the ranks can grow in a cofinal filtration of $\pi_1(N)$.

First note that given any cofinal normal filtration $\{\pi_i\}_{i\in\mathbb{N}}$ of $\pi = \pi_1(N)$ it follows from the work of Lück [20, Theorem 0.1] and Lott and Lück [19, Theorem 0.1] that

$$\lim_{i\to\infty} \frac{1}{[\pi : \pi_i]} b_1(\pi_i) = 0, \tag{1.1}$$

i.e. the first Betti number grows sublinearly. The same equality also holds for almost normal cofinal filtrations of $\pi_1(N)$ if we apply the aforementioned results to an appropriate finite cover of N.

Remark. Note that (1.1) does not necessarily hold for cofinal filtrations of $\pi_1(N)$ which are not almost normal. In fact Girão [8] (see the proof of [8, Theorem 3.1]) gives an example of a cusped hyperbolic 3-manifold together with a cofinal filtration of $\{\pi_i\}_{i\in\mathbb{N}}$ of $\pi = \pi_1(N)$ such that

$$\lim_{i\to\infty} \frac{1}{[\pi : \pi_i]} b_1(\pi_i) > 0.$$

It is an interesting question how quickly $\frac{1}{[\pi:\pi_i]} b_1(\pi_i)$ converges to zero, and to what degree the convergence depends on the choice of normal cofinal filtration of $\pi = \pi_1(N)$. This question for example was recently studied by Kionke and Schwermer [12].

We will use recent work of Agol [2] (which in turn builds on work of Kahn and Markovic [11] and Wise [32]) to prove the following theorem which says that 'most' 3-manifolds admit cofinal filtrations with 'fast' sublinear growth of first Betti numbers.

Theorem 1.1. *Let $N \neq S^1 \times D^2$ and $N \neq T^2 \times I$ be a 3-manifold which is neither spherical nor covered by a torus bundle. Then the following hold*:

(1) *Given any function $f : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ such that*

$$\lim_{n\to\infty} \frac{f(n)}{n} = 0$$

there exists an almost normal cofinal filtration $\{\pi_i\}_{i\in\mathbb{N}}$ of π such that

$$b_1(\pi_i) \geqslant f\big([\pi : \pi_i]\big) \quad \text{for every } i \in \mathbb{N}.$$

(2) *There exists a normal cofinal filtration $\{\pi_i\}_{i\in\mathbb{N}}$ of $\pi = \pi_1(N)$ and an $\varepsilon \in (0, 1)$ such that*

$$b_1(\pi_i) \geqslant [\pi : \pi_i]^{\varepsilon} \quad \text{for every } i \in \mathbb{N}.$$

We now turn to the construction of cofinal filtrations with 'slow' growth of ranks. First note that if H is a finite index subgroup of a finitely generated group G, then it follows from the