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Simultaneous bifurcation of limit cycles from a linear center with extra singular points ☆

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Abstract

The period annuli of the planar vector field $x' = -yF(x,y)$, $y' = xF(x,y)$, where the set $\{F(x,y) = 0\}$ consists of k different isolated points, is defined by $k+1$ concentric annuli. In this paper we perturb it with polynomials of degree n and we study how many limit cycles bifurcate, up to a first order analysis, from all the period annuli simultaneously in terms of k and n. Additionally, we prove that the associated Abelian integral is piecewise rational and, when $k=1$, the provided upper bound is reached. Finally, the case $k=2$ is also treated.

MSC: primary 34C08; secondary 34C07, 37C27

Keywords: Polynomial perturbation of centers; Piecewise rational Abelian integral; Simultaneity of limit cycles from several period annuli

1. Introduction

Let H, f, g be polynomials in x, y such that $\gamma_h \subseteq \{H(x,y) = h\}$, with $h \in (h_0, h_1)$, are simple closed curves around the point $(x_0, y_0) = \gamma_{h_0}$. Then the system

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$$\begin{cases} \dot{x} = \dfrac{\partial H(x,y)}{\partial y} + \varepsilon f(x,y), \\ \dot{y} = -\dfrac{\partial H(x,y)}{\partial x} + \varepsilon g(x,y), \end{cases} \tag{1}$$

has a center at (x_0, y_0) when $\varepsilon = 0$. V.I. Arnol'd, see [1,2], states the *weak Hilbert 16th problem* asking for the maximum number of isolated zeros of the Abelian integral, associated to system (1),

$$I(h) = \oint_{\gamma_h} \big(f(x,y)\,dy - g(x,y)\,dx\big). \tag{2}$$

For this case we have that $I(h)$ is the first order approximation of the Poincaré map. Then, each simple zero of I, h^*, corresponds to a limit cycle of (1) bifurcating from γ_{h^*} for ε small enough, see [17,18]. This function is also known as the *Poincaré–Pontrjagin–Melnikov function* of system (1).

Usually, the centers studied up to now have only one period annulus or, when they have more than one, the study is restricted to one of them. There are not so many papers focused on the study of simultaneous bifurcation of limit cycles from centers with different period annuli. Some of them are [5,8] that deal with the simultaneity in two different regions, or [7] where three separated period annuli appear. A study of the bifurcation of limit cycles from different period annuli of polynomial Hamiltonian systems to obtain lower bounds for the Hilbert number is done in [4] and, more recently, in [14]. In this paper, we consider a center with nested period annuli. The main goals are that we obtain an explicit expression for the Abelian integral and that we can study the number of zeros in all regions simultaneously.

More concretely, the aim of this paper is to study the number of limit cycles that bifurcate, for ε small enough, from system

$$\begin{cases} \dot{x} = yK(x,y) + \varepsilon P(x,y), \\ \dot{y} = -xK(x,y) + \varepsilon Q(x,y), \end{cases} \tag{3}$$

where P and Q are arbitrary real polynomials of degree n and $K(x,y)$ is a specific kind of polynomial. There are several papers where different $K(x,y)$ are considered. In [12,19,20] the set $\{K(x,y) = 0\}$ represents a straight line of simple or multiple singular points. In [3,11,13], the problem when $K(x,y)$ are some concrete quadratic polynomials is considered. When the set $\{K(x,y) = 0\}$ represents a collection of straight lines parallel to one or two orthogonal directions is studied in [9]. Here K is defined by a collection of k different points. We refer to them as the singularities of system (3) or the extra singularities in order to distinguish them from the origin. This work can be considered as a continuation of [10], where only the first period annulus is studied. In [11], among other general conics, the case of one singularity is also done for both period annuli, but only under cubic perturbations. Our aim is to obtain the maximum number of limit cycles that bifurcate from the periodic orbits of all period annuli simultaneously, for a fixed collection of singularities, up to a first order study.

Where $K(x,y)$ does not vanish, after a time rescaling, system (3) is equivalent to

$$\begin{cases} \dot{x} = y + \varepsilon \dfrac{P(x,y)}{K(x,y)}, \\ \dot{y} = -x + \varepsilon \dfrac{Q(x,y)}{K(x,y)}. \end{cases} \tag{4}$$