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Dynamics of non-autonomous parabolic problems with subcritical and critical nonlinearities ☆

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Abstract

This is a subsequent work of our previous one in [25]. Let the nonlinear terms of non-autonomous parabolic problems with singular initial data satisfy subcritical and critical growth conditions. We first establish the existence of uniform attractors in $W^{1,r}(\Omega)$, $1 < r < N$, for the family of processes corresponding to the equations with external forces being translation bounded but not translation compact. Then, we prove the existence of pullback attractors in $L^r(\Omega)$ and $W^{1,r}(\Omega)$, respectively, for the process corresponding to the equation with the weaker assumption on the external force than previous one. Finally, we investigate the robust of attractors and establish the existence of pullback exponential attractors for the process acting in $L^r(\Omega)$ and $W^{1,r}(\Omega)$, respectively.

Keywords: Non-autonomous systems; Parabolic equations; Singular initial data; Attractors

1. Introduction

In this paper, we continue to study the long-time behavior of the following non-autonomous nonlinear parabolic equation

$$\begin{cases} u_t - \Delta u + f(x,u) = g(x,t) & \text{in } \Omega,\ t > \tau, \\ u = 0 & \text{on } \partial\Omega, \\ u(\tau) = u_\tau, & \tau \in \mathbb{R}, \end{cases} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $g(x,t) \in L^p_{loc}(\mathbb{R}; X)$, $p > 1$, X is a Banach space. In order to get the well-posedness of (1.1), we impose some assumptions on nonlinearity:

Assumption I. Suppose that $f(x,u) \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ satisfies

$$f(x,0) = 0, \tag{1.2}$$

$$\big|f(x,u) - f(x,v)\big| \leqslant C\big|a(x)\big||u-v|\big(|u|^{\rho-1} + |v|^{\rho-1} + 1\big) \tag{1.3}$$

with $\rho > 1$ and $a(x) \in L^\beta(\Omega)$, $\beta > 1$.

Under Assumption I and $g(x,t)$ belonging to $L^p_b(\mathbb{R}; L^r(\Omega))$, from the well-posedness of (1.1) with singular initial data in $L^r(\Omega)$ or $W^{1,r}(\Omega)$ established in [25] we know that in critical case, the time of existence of solutions is only uniform on the compact set of initial data spaces. In order to get the uniform existence time of solutions on any bounded set of initial data spaces for (1.1) with critical nonlinearity, we impose the following assumptions on $f(x,u)$:

Assumption II. Suppose that $f(x,u)$ satisfies

$$f(x,u) = a(x)f_1(u) \tag{1.4}$$

with $a(x) \in L^\beta(\Omega)$, $\beta > 1$ and $f_1(s) \in C^1(\mathbb{R})$ such that

$$f_1(0) = 0, \tag{1.5}$$

$$\lim_{|s|\to+\infty} \frac{f_1'(s)}{|s|^{\rho-1}} = 0, \quad \rho > 1. \tag{1.6}$$

In Section 2, under Assumption I or II and $g(x,t)$ belonging to $L^p_{loc}(\mathbb{R}; L^r(\Omega))$, we will establish the well-posedness of (1.1) with singular initial data in $L^r(\Omega)$ and $W^{1,r}(\Omega)$, respectively. For other related study on evolution equations with singular initial data, see, e.g., [2,8,10,12,24,27,31].

When the nonlinearities of (1.1) satisfy Assumption I and subcritical growth conditions, and the external forces are translation bounded in $L^r_{loc}(\mathbb{R}; L^r(\Omega))$ but not translation compact, the existence of uniform attractors for the family of processes acting in $L^r(\Omega)$ or $W^{1,2}(\Omega)$ have been obtained in [25]. The first purpose of this paper is to establish the existence of uniform attractors in $W^{1,r}(\Omega)$, $1 < r < N$ for the family of processes corresponding to (1.1) with subcritical and critical nonlinearities. When the initial data belongs to $L^r(\Omega)$, similar results can be obtained for (1.1) with critical nonlinearities. This will be done in Section 3.

Uniform attractors for the non-autonomous systems are the minimal compact sets which uniformly (w.r.t. time symbol) attract every bounded set of the initial data spaces. Unlike the global attractors for autonomous systems, uniform attractors have not the invariant property, see details in [14]. In order to better understand the dynamics behavior of the non-autonomous evolution equations, pullback attractors are introduced for non-autonomous systems. *Pullback attractor* is a family of compact and invariant sets, and pullback attract every bounded subset of the phase spaces, see, e.g., [13,15,29]. It is noticed that comparing with the existence of uniform attractors for the concrete non-autonomous evolution equations, the existence of pullback attractors for them can be obtained in the weaker assumptions on the external forces. Recently, authors in [9,20,29] extend the concept of pullback attractors and present the pullback $\mathcal{D}$-attractors in consideration of universes of the initial data changing in time which is characterized by a tempered