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The Krull–Gabriel dimension of discrete derived categories

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ABSTRACT

We compute the Krull–Gabriel dimension of the category of perfect complexes for finite dimensional algebras which are derived discrete.

0. Introduction

Let k be an algebraically closed field and Λ a finite dimensional k-algebra. We denote by $\operatorname{mod}\Lambda$ the category of finitely presented Λ-modules and by $\operatorname{proj}\Lambda$ the full subcategory of finitely generated projective Λ-modules.

The Krull–Gabriel dimension of the representation theory of Λ is an invariant first studied by Geigle [11]. For this invariant one considers the abelian category $\mathcal{C} = \operatorname{Ab}(\operatorname{mod}\Lambda)$ of finitely presented functors $\operatorname{mod}\Lambda \to \operatorname{Ab}$ into the category of abelian

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groups. The Krull–Gabriel dimension $\operatorname{KGdim}\mathcal{C}$ of $\mathcal{C}$ is by definition the smallest integer n such that $\mathcal{C}$ admits a filtration by Serre subcategories

$$0 = \mathcal{C}_{-1} \subseteq \mathcal{C}_0 \subseteq \ldots \subseteq \mathcal{C}_n = \mathcal{C},$$

where $\mathcal{C}_i/\mathcal{C}_{i-1}$ is the full subcategory of all objects of finite length in $\mathcal{C}/\mathcal{C}_{i-1}$.

We have $\operatorname{KGdim}\mathcal{C} = 0$ if and only if Λ is of finite representation type by a classical result of Auslander [1], and $\operatorname{KGdim}\mathcal{C} \neq 1$ by a result of Herzog [14] and Krause [16]. In his thesis [11], Geigle proved that $\operatorname{KGdim}\mathcal{C} = 2$, when Λ is tame hereditary.

In this work we investigate the category of perfect complexes which is by definition the bounded derived category $\mathcal{D}^b(\operatorname{proj}\Lambda)$. We compute the Krull–Gabriel dimension of the abelian category $\operatorname{Ab}(\mathcal{D}^b(\operatorname{proj}\Lambda))$, when Λ is derived discrete in the sense of Vossieck [19]. The main result is the following.

Main Theorem. *Let Λ be a finite dimensional k-algebra.*

(1) *If Λ is derived discrete and piecewise hereditary, then*

$$\operatorname{KGdim}\operatorname{Ab}\bigl(\mathcal{D}^b(\operatorname{proj}\Lambda)\bigr) = 0.$$

(2) *If Λ is derived discrete and not piecewise hereditary, then*

$$\operatorname{KGdim}\operatorname{Ab}\bigl(\mathcal{D}^b(\operatorname{proj}\Lambda)\bigr) = \begin{cases} 1 & \textit{if } \operatorname{gl.dim}\Lambda = \infty, \\ 2 & \textit{if } \operatorname{gl.dim}\Lambda < \infty. \end{cases}$$

(3) *If Λ is not derived discrete, then*

$$\operatorname{KGdim}\operatorname{Ab}\bigl(\mathcal{D}^b(\operatorname{proj}\Lambda)\bigr) \geq 2.$$

The rest of this note is devoted to proving this theorem. For an elementary description of the Krull–Gabriel dimension, see Proposition 2.2.

Conventions. By $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{N}_+$, we denote the sets of integers, nonnegative integers, and positive integers, respectively. For $i, j \in \mathbb{Z}$, set

$$[i, j] := \{l \in \mathbb{Z} \mid i \leq l \leq j\}.$$

Furthermore, $[i, \infty) := \{l \in \mathbb{Z} \mid i \leq l\}$ and $(-\infty, j] := \{l \in \mathbb{Z} \mid l \leq j\}$.

1. Derived discrete algebras

Let Λ be a finite dimensional k-algebra. The algebra Λ is called *derived discrete* if for each sequence $(h_n)_{n\in\mathbb{Z}}$ of nonnegative integers there are only finitely many isomorphism