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The principal frequency of Δ_∞ as a limit of Rayleigh quotients involving Luxemburg norms

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Abstract

The asymptotic behavior of Rayleigh quotients involving both Luxemburg norms and modulars in the variable exponent Lebesgue space $L^{p(\cdot)}$ is studied as $p(\cdot) \to \infty$. In a particular case, we recover a well-known result of Juutinen, Lindqvist and Manfredi regarding the limit, as $p \to \infty$ of the minima of Rayleigh quotients associated to the eigenvalue problem for the p-Laplacian.

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1. Introduction and main results

Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set with smooth boundary. Throughout this paper we denote by $\mathcal{P}(\Omega)$ the class of variable exponents $p : \Omega \to (1, \infty)$ that are continuous and satisfy $p^- := \inf_{x\in\Omega} p(x) > 1$ and $p^+ := \sup_{x\in\Omega} p(x) < +\infty$. Let $\{p_n\} \subset \mathcal{P}(\Omega)$ be a sequence of functions in $C^1(\Omega)$ such that

$$p_n^- \to \infty \quad \text{as } n \to \infty, \tag{1.1}$$

$$\text{there exists a real constant } \beta > 1 \quad \text{such that} \quad p_n^+ \leqslant \beta p_n^- \quad \text{for all } n \in \mathbb{N}, \tag{1.2}$$

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and

$$\lim_{n\to\infty}\frac{|\nabla p_n(x)|}{p_n(x)^2}=0, \quad \forall x\in\Omega. \tag{1.3}$$

Note that (1.2) implies in particular that

$$p_n\to\infty \quad \text{uniformly in } \Omega. \tag{1.4}$$

We also remark that both (1.2) and (1.3) hold if we assume, for example, that there exists a function $\xi\in C(\Omega)$ such that

$$\nabla \ln p_n(\cdot)\to\xi \quad \text{uniformly in } \Omega. \tag{1.5}$$

Conditions of type (1.4) and (1.5) were initially introduced in [11] in the context of a study of the asymptotic behavior of $p_n(\cdot)$-harmonic functions as $p_n\to\infty$, while conditions such as (1.1) and (1.2) appear initially in [1].

In what follows $|v|_{q(\cdot)}$ will denote the Luxemburg norm of v in the variable exponent Lebesgue space $L^{q(\cdot)}(\Omega;\mathbb{R}^m)$, $m\in\mathbb{N}$. We refer to Section 2 of the paper for the precise definitions, as well as for more details on variable exponent Lebesgue and Sobolev spaces.

Definition 1. We say that $p\in\mathcal{P}^{\log}(\Omega)$ if $p\in\mathcal{P}(\Omega)$ and, in addition, p satisfies the global log-Hölder continuity condition: there exist $c_1, c_2>0$, and $p_\infty\in\mathbb{R}$ such that

$$\left|p(x)-p(y)\right|\leqslant\frac{c_1}{\ln(e+1/|x-y|)} \quad \text{for all } x,y\in\Omega,$$

and

$$\left|p(x)-p_\infty\right|\leqslant\frac{c_2}{\ln(e+|x|)}, \quad \text{for all } x\in\Omega.$$

Remark 1. Typical examples of sequences of functions $p_n\in\mathcal{P}^{\log}(\Omega)$ that satisfy our assumptions (1.1), (1.2), and (1.3) are: $p_n(x)=n$, $p_n(x)=p(x)+n$, $p_n(x)=np(x/n)$, and $p_n(x)=np(x)$, where $p\in\mathcal{P}^{\log}(\Omega)$. These examples are taken from [11]. In particular, (1.5) also holds, with $\xi\equiv 0$ in all but the last example, in which case one has $\xi=\nabla\ln p$.

Let $\delta:\Omega\to[0,\infty)$ be the distance function to $\partial\Omega$, given by $\delta(x):=\operatorname{dist}(x,\partial\Omega)=\inf_{y\in\partial\Omega}|x-y|$. It is known (see Lemma 1.5 and Section 2 in [9]) that if one defines

$$\Lambda_\infty:=\inf\left\{\frac{\|\nabla\varphi\|_{L^\infty(\Omega)}}{\|\varphi\|_{L^\infty(\Omega)}}:\ \varphi\in W_0^{1,\infty}(\Omega)\setminus\{0\}\right\}, \tag{1.6}$$

then the infimum above is always achieved at δ, that is,

$$\Lambda_\infty=\frac{\|\nabla\delta\|_{L^\infty(\Omega)}}{\|\delta\|_{L^\infty(\Omega)}}=\frac{1}{\max\{\operatorname{dist}(x,\partial\Omega):\ x\in\Omega\}},$$

and that we have

$$\lim_{p\to\infty}\inf\left\{\frac{\|\nabla\varphi\|_{L^p(\Omega)}}{\|\varphi\|_{L^p(\Omega)}}:\ \varphi\in W_0^{1,p}(\Omega)\setminus\{0\}\right\}=\Lambda_\infty. \tag{1.7}$$

This last identity justifies the name of "principal frequency" (of Δ_∞) given in [9] to Λ_∞. Here, Δ_∞ stands for the ∞-Laplace operator, which on smooth functions u is defined by