Available online at www.sciencedirect.com

ScienceDirect

Bull. Sci. math. 138 (2014) 253–269

BULLETIN DES SCIENCES MATHÉMATIQUES

www.elsevier.com/locate/bulsci

Brennan's Conjecture and universal Sobolev inequalities

V. Gol'dshtein *, A. Ukhlov

Department of Mathematics, Ben-Gurion University of the Negev, P.O. Box 653, Beer-Sheva, 84105, Israel

Received 9 June 2013

Available online 1 July 2013

Abstract

Brennan's Conjecture states integrability of derivatives of plane conformal homeomorphisms $\varphi : \Omega \to \mathbb{D}$ that map a simply connected plane domain with non-empty boundary $\Omega \subset \mathbb{C}$ to the unit disc $\mathbb{D} \subset \mathbb{R}^2$. We prove that Brennan's Conjecture leads to existence of compact embeddings of Sobolev spaces $\overset{\circ}{W}{}^1_p(\Omega)$ into weighted Lebesgue spaces $L_q(\Omega, h)$ with universal conformal weights $h(z) := J(z,\varphi) = |\varphi'(z)|^2$. For $p = 2$ the number q is an arbitrary number between 1 and ∞ (Gol'dshtein and Ukhlov, in press [12]), for $p \neq 2$ the number q depends on p and the integrability exponent for Brennan's Conjecture.

MSC: 46E35; 30C65

Keywords: Sobolev inequalities; Conformal mappings; Brennan's Conjecture; Elliptic equations

1. Introduction

Let $\Omega \subset \mathbb{C}$ be a simply connected plane domain with non-empty boundary. We study the weighted Poincaré–Sobolev inequalities

$$\left(\int_{\Omega} |f(z)|^r h(z)\, d\mu\right)^{\frac{1}{r}} \leqslant K\left(\int_{\Omega} |\nabla f(z)|^p \, d\mu\right)^{\frac{1}{p}} \tag{1.1}$$

for functions f of the Sobolev space $\overset{\circ}{W}{}^1_p(\Omega)$ and special weights $h(z) := J(z,\varphi) = |\varphi'(z)|^2$ induced by conformal homeomorphisms $\varphi : \Omega \to \mathbb{D}$. Recall that $\overset{\circ}{W}{}^1_p(\Omega)$ is closure of the set

* Corresponding author. Tel.: +972 8 6461620.
E-mail addresses: vladimir@math.bgu.ac.il (V. Gol'dshtein), ukhlov@math.bgu.ac.il (A. Ukhlov).

0007-4497/$ – see front matter
http://dx.doi.org/10.1016/j.bulsci.2013.06.005

of all smooth functions with compact support in Ω in the Sobolev space $W^1_p(\Omega)$. The number r depends on p and does not depends on choice of Ω. By this reason we call these inequalities "universal". Its "universality" based on the Riemann Mappings Theorem and the Brennan's Conjecture interpretation in terms of composition operators [11].

The weight $h(z)$ can be defined also in the terms of the hyperbolic geometry of Ω and $\mathbb{D}$. Namely

$$h(z) = \frac{\lambda^2_{\Omega}(z)}{\lambda^2_{\mathbb{D}}(\varphi(z))}$$

when $\lambda_{\mathbb{D}}$ and λ_{Ω} are hyperbolic metrics in $\mathbb{D}$ and Ω [1].

Note, that two different conformal homeomorphisms $\varphi : \Omega \to \mathbb{D}$ and $\tilde{\varphi} : \Omega \to \mathbb{D}$ can be connected by a conformal automorphism $\eta : \mathbb{D} \to \mathbb{D}$ (i.e., $\varphi = \tilde{\varphi} \circ \eta$), the conformal weights induced by φ and $\tilde{\varphi}$ are equivalent. It means that $h(z) = J(z, \varphi) \lesssim J(z, \tilde{\varphi}) \lesssim J(z, \varphi) = h(z)$, and therefore the weighted Lebesgue space $L_q(\Omega, h)$ does not depend on the choice of a conformal homeomorphism and depends only on the conformal structure (hyperbolic geometry) of Ω.

Here we have used the following notations: $z = x + iy$ *is a complex number*, $f(z) = f(x, y)$ *is a real-valued function*, $\nabla f(z) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ *is the weak gradient of* f, μ *is the Lebesgue measure*.

Brennan's Conjecture [3] is that for a conformal homeomorphism $\varphi : \Omega \to \mathbb{D}$

$$\int_{\Omega} |\varphi'(z)|^s \, d\mu < +\infty, \quad \text{for all } \frac{4}{3} < s < 4. \tag{1.2}$$

For $4/3 < s < 3$, it is a comparatively easy consequence of the Koebe distortion theorem (see, for example, [2]). J. Brennan [3] (1973) extended this range to $4/3 < s < 3 + \delta$, where $\delta > 0$, and conjectured it to hold for $4/3 < s < 4$. The example of $\Omega = \mathbb{C} \setminus (-\infty, -1/4]$ shows that this range of s cannot be extended. The upper bound of those s for which (1.2) is known to hold has been increased to $s \leqslant 3.399$ by Ch. Pommerenke, to $s \leqslant 3.421$ by D. Bertilsson, and then to $s \leqslant 3.752$ by Hedenmalm and Shimorin (2005). These results and more information can be found in [2,19,20].

A connection between Brennan's Conjecture and composition operators was established in [11]:

Theorem 1.1 *(Equivalence Theorem). Brennan's Conjecture* (1.2) *holds for a number* $s \in (4/3; 4)$ *if and only if any conformal homeomorphism* $\varphi : \Omega \to \mathbb{D}$ *induces a bounded composition operator*

$$\varphi^* : L^1_p(\mathbb{D}) \to L^1_{q(p,s)}(\Omega)$$

for any $p \in (2; +\infty)$ *and* $q(p, s) = ps/(p + s - 2)$.

Remark. Brennan's Conjecture is proved for some special classes of domains: starlike domains, bounded domains which boundaries are locally graphs of continuous functions, etc.

For conformal homeomorphisms $\psi : \mathbb{D} \to \Omega$, Brennan's Conjecture can be reformulated as the Inverse Brennan's Conjecture

$$\int_{\mathbb{D}} |\psi'(w)|^{\alpha} \, d\mu < +\infty, \quad \text{for all } -2 < \alpha < 2/3$$

where $\alpha = 2 - s$.