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On the time inhomogeneous skew Brownian motion ☆

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Abstract

This paper is devoted to the construction of a solution for the "Inhomogeneous skew Brownian motion" equation, which first appeared in a seminal paper by Sophie Weinryb, and recently, studied by Étoré and Martinez. Our method is based on the use of the Balayage formula. At the end of this paper we study a limit theorem of solutions.

MSC: 60H10; 60J60

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1. Introduction

The skew Brownian motion appeared in the 70s in the seminal work [8] of Itô and McKean as a natural generalization of the Brownian motion. It is a process that behaves like a Brownian motion except that the sign of each excursion is chosen using an independent Bernoulli random variable of parameter α.

As shown in [7], this process is a strong solution to some stochastic differential equation (SDE) with singular drift coefficient:

$$X_t = x + B_t + (2\alpha - 1)L_t^0(X) \tag{1}$$

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where $\alpha \in (0, 1)$ is the skewness parameter, $x \in \mathbb{R}$, and $L_t^0(X)$ stands for the symmetric local time at 0.

The reader may find many references concerning the homogeneous skew Brownian motion and various extensions in the literature. We cite Walsh [17], Harisson and Shepp [7], LeGall [9] and Ouknine [13].

A related stochastic differential equation, introduced by Weinryb [19] is:

$$X_t^\alpha = x + B_t + \int_0^t \big(2\alpha(s) - 1\big)\, dL_s^0\big(X^\alpha\big), \quad t \geqslant 0, \tag{2}$$

where $(B_t)_{t\geqslant 0}$ is a standard Brownian motion on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\alpha : \mathbb{R}^+ \to [0, 1]$ is a Borel function and $L^0(X^\alpha)$ stands for the symmetric local time at 0 of the unknown process X^α.

The process X^α will be called "time inhomogeneous skew Brownian motion" (ISBM in short). Of course, this equation is an extension of the skew Brownian motion.

In [19], it was shown that there is a pathwise uniqueness for Eq. (2) but in [19] the local time appearing in the equation is standard right sided local time, so that the function α is supposed to take values in $]-\infty, 1/2]$. As is well known, weak existence combined with pathwise uniqueness, establishes existence and uniqueness of a strong solution to (2), via the classical result of Yamada and Watanabe [18]. So, our purpose in this paper, is to give an explicit construction of the solution of (2) by approximating the function α by a sequence of piecewise constant functions (α_n). In order to treat the simple case of a given piecewise constant, we are inspired by a construction given by Étoré and Martinez [5], but our proof is totally different. Instead of trying to show that our construction preserves the Markov property and that the constructed process satisfies (2), we use the Balayage formula: the key of "first order calculus". After its first appearance in Azéma and Yor [1], it was later studied extensively in a series of papers as [4,16,20]. Note that the point of our departure in this sense, is an interesting observation of Prokaj (see [14, Proposition 3]), his work is strongly related to the work of Gilat [6].

In [6], the author proved that every nonnegative submartingale is equal in law to the absolute value of a martingale M. Barlow in [2] gives an explicit construction of the martingale M but for a remarkable class of submartingales. We will show that this result of Barlow is a direct consequence of the Balayage formula.

The paper is organized as follows: The second section, we start it with the progressive version of the Balayage formula and we show how to deduce from it a generalization of the observation of Prokaj, in the same section we give a simple proof of the result of Barlow [2]. Section 3 is devoted to the construction of a weak solution of Eq. (2) with a piecewise constant function. Extension of the above result to a general case where α is a Borel function is the subject of the same section. At the end of this work, we study the stability of the solutions of Eq. (2) by using the Skorokhod representation theorem. This result was obtained by Étoré and Martinez [5] but under some monotonicity assumptions.

1.1. Preliminaries

The ISBM has many interesting and sometimes unexpected properties, see Étoré and Martinez [5]. The main facts that we use in this paper will be summarized in this section.