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A uniform estimate for rough paths

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Abstract

It is well known that for two p-rough paths, if their first $\lfloor p \rfloor$ levels of iterated integrals are close in p-variation sense, then all levels of their iterated integrals are close. In this paper, we prove that a similar result holds for the paths provided the first $\lfloor p \rfloor$ terms are close in a 'uniform' sense. The estimate is explicit, dimension free, and only involves the p-variation of two paths and the 'uniform' distance between the first $\lfloor p \rfloor$ terms. Applications include estimation of the difference of the signatures of two uniformly close paths (Lyons and Xu, 2011 [6]), and convergence rates for Gaussian rough paths (Riedel and Xu, 2012 [7]).
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1. Introduction

1.1. Motivation

The classical continuity theorem in rough paths (Theorem 2.2.2 in [4]) states that if X and Y are two p-rough paths whose p-variation are both controlled by ω and such that

$$\left\| X_{s,t}^k - Y_{s,t}^k \right\| \leqslant \epsilon \frac{\omega(s,t)^{\frac{k}{p}}}{\beta(\frac{k}{p})!}, \quad \forall k = 1, \dots, \lfloor p \rfloor, \tag{1}$$

then (1) holds for all $k \geqslant 1$. The proof is by an induction argument, which depends on the value of the exponent on the control, namely $\frac{k}{p}$. Although it is powerful in many places, there are certain problems for which we need a more convenient (and weaker) assumption. More precisely, we assume

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$$\left\| X_{s,t}^{k} - Y_{s,t}^{k} \right\| < \epsilon \frac{\omega(s,t)^{\frac{k-\delta}{p}}}{\beta(\frac{k}{p})!}, \quad \forall k = 1, \ldots, \lfloor p \rfloor, \tag{2}$$

where $\delta \in [0, 1]$. We wish to study whether similar estimates hold for $k \geqslant \lfloor p \rfloor + 1$. It is easy to see that the classical assumption (1) corresponds to $\delta = 0$.

Such estimates are useful in a number of problems. For example, consider the following two linear differential equations

$$dx_t = Ax_t\, d\gamma_t, \qquad x_0 = a, \tag{3}$$

and

$$dy_t = Ax_t\, d\tilde{\gamma}_t, \qquad y_0 = a, \tag{4}$$

where $\gamma, \tilde{\gamma} : [0, 1] \to \mathbb{R}^d$ are two paths of bounded variations whose lengths are both controlled by ω. Suppose further that

$$\sup_{t \in [0,1]} |\gamma_t - \tilde{\gamma}_t| < \epsilon, \tag{5}$$

and one wishes to estimate the difference of the solution flow $|x_t - y_t|$. This question involves estimating the differences between all higher degrees of iterated integrals of γ and $\tilde{\gamma}$, which are called signatures (we will give a precise definition in the next section). If we let X and Y be the signatures of γ and $\tilde{\gamma}$, then assumption (5) can be written as

$$\left\| X_{s,t}^{1} - Y_{s,t}^{1} \right\| \leqslant 2\epsilon = 2\epsilon\omega(s,t)^{0}.$$

We see that it falls in the assumption (2) with $p = 1$ and $\delta = 1$. We will come back to this question at the end of this paper.

Our estimates also apply to obtaining the convergence rates of Gaussian rough paths. For details, we refer to the recent works [1] and [7].

Notation. In what follows, p will always be a number that is at least 1. We use $\lfloor p \rfloor$ to denote the largest integer that does not exceed p, and let $\{p\} = p - \lfloor p \rfloor$ be the fractal part of p.

1.2. Main results

Before stating our main result, let us explain briefly why the induction argument for the classical continuity theorem does not work directly here. As mentioned earlier, the induction argument depends on the exponent $\frac{n}{p}$. More precisely, the exponent for the level $n + 1 = \lfloor p \rfloor + 1$ is expected to be

$$\frac{\lfloor p \rfloor + 1}{p} > 1. \tag{6}$$

This ensures that when one repeats Young's trick of dropping points, the total sum will converge. However, this condition is not satisfied in our problem (2) unless $\delta < 1 - \{p\}$.

To this point, one may wonder whether one can immediately get the estimate by raising the control to an appropriate power so that the new control satisfies assumption (1). Unfortunately this does not work, for there is no fixed power that one can do it in a homogeneous way for all $k \leqslant \lfloor p \rfloor$. Furthermore, the new control will in general fail to be superadditive.

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