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A probabilistic model associated with the pressureless gas dynamics

Sergio Albeverio [a,b,c,d], Anastasia Korshunova [e], Olga Rozanova [e,*]

[a] *Universität Bonn, Institut für Angewandte Mathematik, Abteilung für Stochastik, Endenicher Allee 60, D-53115 Bonn, Germany*
[b] *HCM, SFB611 and IZKS, Bonn, Germany*
[c] *BiBoS, Bielefeld–Bonn, Germany*
[d] *CERFIM, Locarno, Switzerland*
[e] *Department of Mechanics and Mathematics, Moscow State University, Moscow 119992, Russia*

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Abstract

Using a method of stochastic perturbation of a Langevin system associated with the non-viscous Burgers equation we introduce a system of PDE that can be considered as a regularization of the pressureless gas dynamics describing sticky particles. By means of this regularization we describe how starting from smooth data a δ-singularity arises in the component of density. Namely, we find the asymptotics of solution at the point of the singularity formation as the parameter of stochastic perturbation tends to zero. Then we introduce a generalized solution in the sense of free particles (FP-solution) as a special limit of the solution to the regularized system. This solution corresponds to a medium consisting of non-interacting particles. The FP-solution is a bridging step to constructing solutions to the Riemann problem for the pressureless gas dynamics describing sticky particles. We analyze the difference in the behavior of discontinuous solutions for these two models and the relations between them. In our framework we obtain a unique entropy solution to the Riemann problem in 1D case.

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Keywords: Pressureless gas; Non-viscous Burgers equation; Stochastic perturbation; Non-interacting particles; Sticky particles; δ-Singularity; Hugoniot conditions; Spurious pressure

* Corresponding author.
E-mail address: rozanova@mech.math.msu.su (O. Rozanova).

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0. Introduction

The system of pressureless gas dynamics consists of two equations for the components of the density f and velocity u expressing the conservation of mass and momentum:

$$\partial_t f + \operatorname{div}_x(fu) = 0, \qquad \partial_t(fu) + \nabla_x(fu \otimes u) = 0. \tag{1}$$

This system is very important since it is believed to be the simplest model describing the formation of structures in the universe (e.g. [30]) and plays a significant role in the theory of cooling gases and granular materials [9].

System (1) first appears to be very simple, however a closer analysis reveals that it has some peculiar features due to its non-strict hyperbolicity. The system has attracted a significant interest in the last decades and has been investigated quite intensively. In particular, it is well known that the arising in the velocity component of unbounded space derivatives implies the generation of a δ-singularity in the component of the density. Therefore for this system one needs to define a generalized or measure-valued solution of a special kind. This was done in [21,8,17,16,32,25,10], where the authors used different techniques (vanishing viscosity, weak asymptotics, variational principle, duality) to define the solution and prove its existence. Further, the Riemann problem for the pressureless gas dynamics was studied (e.g. [32,17,31]), including a singular Riemann problem with a δ-singularity concentrated at the jump at the initial moment. Nevertheless, even in the 1D case there are certain open problems, not to mention those present in the higher dimensional situation. In particular, there is a problem concerning the uniqueness of solutions. Both in the case of rarefaction and contraction it is possible to construct a whole family of solutions to the Riemann problem satisfying the integral identities and entropy conditions that are used to single out the unique solution in the strictly hyperbolic case (see [13,20] for details). Further, the process of singularity formation was described up to now only in a very special situation [14,15].

We propose to study properties of the system (1) by means of an intermediate integro-differential system obtained from a stochastic perturbation of a Langevin system associated with the non-viscous Burgers equation

$$\partial_t u + (u, \nabla)u = 0. \tag{2}$$

The vectorial equation (2) follows from (1) provided the solution to (1) is smooth. The above-mentioned integro-differential system depends on a parameter of stochastic perturbation σ and can be considered as a regularization of the pressureless gas dynamics. In particular, if the solution to the system of the pressureless gas dynamics system is smooth, the respective solution to the regularizing system tends to it as $\sigma \to 0$. Therefore our method allows to study asymptotics of a smooth solution to the pressureless gas dynamics at the point of a singularity formation in terms of the parameter of regularization σ. Moreover, by means of this method one can find a unique solution to the Riemann problem with arbitrary smooth left and right states. The most clear and explicit results concern the case of constant left and right states. As a bridging step we introduce a new type of generalized solution for the pressureless gas dynamics model, the solution in the sense of free particles (FP-solution). In the case of initial data corresponding to a compression wave the FP-solution is a solution of a gas dynamic system with a specific pressure term.

The paper is organized as follows. In Section 1 we consider the model of motion for free particles perturbed along their trajectories and introduce the integral characteristics of the medium consisting of these particles, in particular, the mean velocity at a fixed coordinate. In Section 2