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Conditions of hyperbolicity of linear differentiable systems with constant multiplicity

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Abstract

Let h be a system with characteristics of constant multiplicity. We prove that if there exists an operator $\mathcal{A}'$ such that $h \circ \mathcal{A}'$ has diagonal principal part and admits a good decomposition, then h must satisfy the Levi conditions.

MSC: 35L45

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1. Introduction

Let $x = (x_0, x') = (x_0, x_1, \dots, x_n) \in \Omega$, Ω neighborhood of $0 \in \mathbb{R}^{n+1}$, we consider an $N \times N$ linear first order system of differential operators

$$h(x, D) = a(x, D) + b(x),$$

where $D = (D_0, D') = (D_0, D_1, \dots, D_n)$, $D_0 = \frac{\partial}{\partial x_0}$, $D_j = \frac{\partial}{\partial x_j}$. $a(x, \xi)$ is the principal symbol of h, $\xi = (\xi_0, \xi') = (\xi_0, \xi_1, \dots, \xi_n)$. a and b are $N \times N$ matrices with analytic coefficients.

We consider the Cauchy problem for h:

$$\begin{cases} h(x, D)u(x) = f(x), \\ u|_{x_0 = \underline{x_0}} = u_0(x'). \end{cases} \tag{1}$$

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Definition 1. h is hyperbolic means that the Cauchy problem (1) is uniformly well-posed in $\mathcal{C}^\infty(\Omega)$.

Let $\mathcal{O}[\xi]$ be the ring of homogeneous polynomials in ξ, with coefficients from the ring of analytic germs in x at $x = 0$, and let $\mathcal{M}_N(\mathcal{O}[\xi])$ be the set of the $N \times N$ matrix, whose entries belong to $\mathcal{O}[\xi]$. In $\mathcal{O}[\xi]$ we have the decomposition

$$\det a(x;\xi) = H_1^{m_1}(x;\xi)\cdots H_{\tau_0}^{m_{\tau_0}}(x;\xi), \tag{2}$$

where H_τ, $\tau = 1,\dots,\tau_0$, are irreducible polynomials, homogeneous of degree s_τ in ξ, with analytic coefficients in x and $m,\dots,m_{\tau_0} \in \mathbb{N}$ do not depend on $(x,\xi) \in \Omega \times \mathbb{R}^n \setminus \{0\}$.

We assume that $\det a(x;\xi)$ is a hyperbolic polynomial of *constant multiplicity*: the polynomial $H_1 \cdots H_{\tau_0}$ is strictly hyperbolic with respect to $(1,0,\dots,0)$ for any $x \in \Omega$, i.e. the solutions in ξ_0 of the equation

$$H_1\bigl(x;\xi_0,\xi'\bigr)\cdots H_{\tau_0}\bigl(x;\xi_0,\xi'\bigr) = 0$$

are real and distinct for any $(x,\xi') \in \Omega \times \mathbb{R}^n \setminus \{0\}$. This assumption is equivalent to the following decomposition

$$\det a(x;\xi) = \prod_{j=1}^{r}\bigl(\xi_0 - \lambda_{(j)}\bigl(x;\xi'\bigr)\bigr)^{m_{(j)}},$$

where the $\lambda_{(j)}$ are real analytic functions with

$$\inf_{\substack{x\in\Omega,\,|\xi'|=1\\ j\neq k}} \bigl|\lambda_{(j)}\bigl(x;\xi'\bigr) - \lambda_{(k)}\bigl(x;\xi'\bigr)\bigr| \neq 0,$$

and the $m_{(j)}$ are constant integers (see [10]).

To simplify the presentation, in the following we assume that in (2) there is only one multiple factor H, of degree s and multiplicity m, and a simple factor K, of degree χ, but the general case can be treated in a similar way, or, equivalently:

$$\begin{aligned}\det a(x;\xi) = {} & \bigl(\xi_0 - \lambda_{(1)}\bigl(x;\xi'\bigr)\bigr)^m \cdots \bigl(\xi_0 - \lambda_{(s)}\bigl(x;\xi'\bigr)\bigr)^m \\ & \times \bigl(\xi_0 - \lambda_{(s+1)}\bigl(x;\xi'\bigr)\bigr)\cdots\bigl(\xi_0 - \lambda_{(s+\chi)}\bigl(x;\xi'\bigr)\bigr).\end{aligned} \tag{3}$$

We consider the problem: what are the conditions on a and b in order that h is hyperbolic?

We have previously defined the conditions L [15]. Before to state them, we recall some notations.

Let (H) be the prime ideal of $\mathcal{O}[\xi]$ defined by H, we consider $\mathcal{O}[\xi]/_{(H)}$, the localized ring of $\mathcal{O}[\xi]$ with respect to (H). $\mathcal{O}[\xi]/_{(H)}$ is a principal ring, whose elements are the fractions whose denominators are not divisible by H; its ideals are generated by the powers of H. In $\mathcal{O}[\xi]/_{(H)}$ the matrix $a(x,\xi)$ is equivalent to the diagonal matrix:

$$\operatorname{diag}\bigl[H^p, H^{q_1},\dots,H^{q_\ell},1,\dots,1\bigr],$$

where $p, q_1,\dots,q_\ell$ are such that:

$$p \geqslant q_1 \geqslant \cdots \geqslant q_\ell > 0, \qquad p + q = m, \qquad q := q_1 + \cdots + q_\ell.$$

The sequence $(p, q_1,\dots,q_\ell)$ will be called the *type* of the operator.