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# A system of fifth-order partial differential equations describing a surface which contains many circles

Kiyoomi Kataoka [a,*,1], Nobuko Takeuchi [b]

[a] *Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan*

[b] *Department of Mathematics, Tokyo Gakugei University, Koganei-shi, Tokyo 184-8501, Japan*

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**Abstract**

Let $z = f(x, y)$ be a germ of a $C^5$-surface at the origin in $\mathbb{R}^3$ containing several continuous families of circular arcs. For examples, a usual torus with 4 such families and Blum cyclides with 6 such families, which are special cases of Darboux cyclides. We introduce a system of fifth-order nonlinear partial differential equations for $f$, and prove that this system describes such a surface germ completely. As applications, we obtain the analyticity of $f$, the finite dimensionality of the solution space of such a system of differential equations with an upper estimate 21 for the dimension. Further we obtained some local characterization of Darboux cyclides by using this system of equations in our forthcoming paper: K. Kataoka, N. Takeuchi, The non-integrability of some system of fifth-order partial differential equations describing surfaces containing 6 families of circles, RIMS Kokyuroku Bessatsu Kyoto University, in press [1].


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* Corresponding author.
*E-mail addresses:* kiyoomi@ms.u-tokyo.ac.jp (K. Kataoka), nobuko@u-gakugei.ac.jp (N. Takeuchi).

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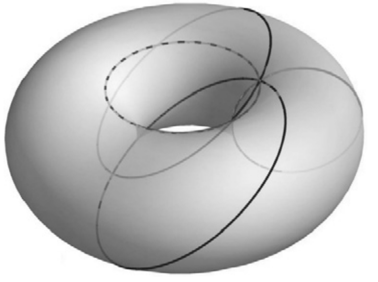

Fig. 1. A torus and Villarceau circles.

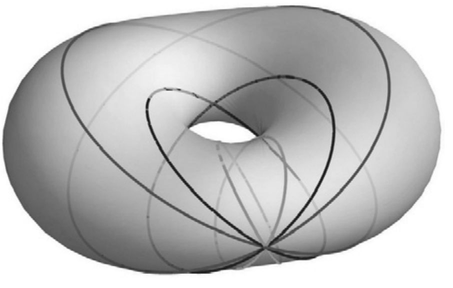

Fig. 2. An example of Blum cyclides: $(x^2 + y^2 + z^2)^2 - 6x^2 - 4y^2 + 4z^2 + 1 = 0$.

## 1. Introduction

In 1848, Yvon Villarceau [2] found that a usual torus includes 4 continuous families of circles passing through every point of the surface; of course, only two of them are new. These new circles, so called Villarceau circles, are slanted against the rotation axis and are not perpendicular to this axis (see Fig. 1). Further in 1980, Richard Blum [3] found that some special cyclides include 4–6 continuous families of circles passing through every point of them (see Fig. 2). Here, a general cyclide due to Darboux [4] is defined by a quartic equation

$$\alpha\left(x_1^2 + x_2^2 + x_3^2\right)^2 + 2\left(x_1^2 + x_2^2 + x_3^2\right)\sum_{i=1}^{3}\beta_i x_i + \sum_{i,j=1}^{3}\gamma_{ij}x_i x_j + 2\sum_{i=1}^{3}\delta_i x_i + \epsilon = 0 \qquad (1.1)$$

with real numbers $\alpha \neq 0$, $\beta_i$, $\gamma_{ij}$, $\delta_i$, $\epsilon$. Then a usual torus and a 6-circle Blum cyclide correspond to the case $\alpha = 1$, $\beta_* = 0$, $\delta_* = 0$, $\gamma_{ij} = -2a_i\delta_{ij}$, $\epsilon = a_4^2$ with $0 < a_4 < a_1 = a_2$, $a_3 = -a_4$, and to that with $0 < a_4 < a_2 < a_1$, $-a_4 \neq a_3 < a_4$, respectively. At the same time, R. Blum gave the following conjecture in [3]:

**Conjecture 1.** *A closed $C^\infty$-surface in $\mathbb{R}^3$ which contains seven circles through each point is a sphere.*

N. Takeuchi [5] (and [6]: the survey of her results) solved this conjecture affirmatively for closed surfaces with genus $g \leqslant 1$ by using the intersection number theory for 1-dimensional homotopy groups. Further, replacing 1-dimensional homotopy groups by 1-dimensional homology groups with $\mathbb{Z}_2$-coefficients, we obtained the following extension in [7]: