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Higher order approximation of complex analytic sets by algebraic sets ☆

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ABSTRACT

Let X be any locally analytic subset of $\mathbf{C}^m$. We show that for every a in X and for every natural number ν, there is an algebraic subset X_ν of $\mathbf{C}^m$ approximating X, in some neighborhood of a, such that the order of tangency of X and X_ν at a is at least ν.

1. Introduction

A basic question in analytic geometry is whether every analytic set Y can be approximated by algebraic ones. In the local version of this problem it is natural to require that approximating algebraic sets are tangent to Y at a fixed point with any prescribed order of tangency.

The latter problem has been recently studied for real analytic (or more generally for semi-analytic and subanalytic) sets by M. Ferrarotti, E. Fortuna and L. Wilson

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(see [16–18]). In [16] and [18] the authors have shown that every closed semi-analytic set of positive codimension admits local real algebraic approximations of any order. (There exist subanalytic sets which do not admit higher order algebraic approximations (cf. [17]). However, in the topology of local uniform convergence, also called the Kuratowski convergence, subanalytic sets can be approximated by semi-algebraic ones as shown by Z. Denkowska and M. Denkowski (see [15]).)

In complex geometry higher order approximations of complex analytic sets by algebraic ones were discussed by R.W. Braun, R. Meise and B.A. Taylor in [10] and [11]. Given a purely dimensional complex analytic set $X \subseteq \mathbf{C}^m$ containing $0 \in \mathbf{C}^m$, the authors have constructed algebraic higher order tangents to $X \cap \Gamma$ at 0, where Γ is a conoid around some curve, whose vertex is 0. These results were used to derive Phragmén–Lindelöf conditions for analytic varieties (see [11] and references therein).

The local version of our main result says that for X as above there is an open neighborhood U of 0 in $\mathbf{C}^m$ such that $X \cap U$ can be approximated by algebraic sets with any prescribed order of tangency at 0. This is a complex analogue of the main theorem of [18] (but the methods we use are completely different from [18] where semi-algebraic geometry techniques are employed). Proving this result we strengthen the main theorems of [5] and [6] which are the starting point for considerations in the present paper. (In [5,6] it is shown that complex analytic sets can be approximated by Nash sets with any prescribed order of tangency, and that complex analytic sets with proper projection are limits of sequences of algebraic sets converging in the sense of holomorphic chains.)

Let $dist(\cdot,\cdot)$ denote the Hausdorff distance between relatively compact subsets of $\mathbf{C}^m$. For any $r > 0$ and any subset $U \subset \mathbf{C}^m$, define $U \cdot r = \{x \in \mathbf{C}^m : \frac{1}{r}x \in U\}$. The notion of the convergence of analytic sets in the sense of chains is introduced in Section 2.

Theorem 1.1. *Let X be an analytic subset of some open set in $\mathbf{C}^m$ of pure dimension n such that X contains $\mathbf{0} \in \mathbf{C}^m$. Then there are an open neighborhood U of $\mathbf{0}$ in $\mathbf{C}^m$ and a sequence $\{X_\nu\}$ of algebraic subsets of $\mathbf{C}^m$ of pure dimension n such that $\{X_\nu \cap U\}$ converges to $X \cap U$ in the sense of chains. Moreover, $dist(X \cap (U \cdot r), X_\nu \cap (U \cdot r)) \leq r^\nu$, for every $0 < r < 1$, $\nu \in \mathbf{N}$.*

In fact in Section 3 we prove a stronger semi-global Theorem 3.2 which immediately implies Theorem 1.1. This is because every purely n-dimensional analytic set is locally an analytic cover, i.e. it is with proper projection onto an open ball in some n-dimensional linear subspace of $\mathbf{C}^m$.

Finally, let us mention that approximation by algebraic sets is closely related to the classical problem of transforming subsets of $\mathbf{C}^m$ or $\mathbf{R}^m$ onto algebraic sets, studied by several authors (see [1–3,8,9,14,20,21,23–29,34], see also [7], p. 24).