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Equivalent Harnack and gradient inequalities for pointwise curvature lower bound ☆

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Abstract

By using a coupling method, an explicit log-Harnack inequality with local geometry quantities is established for (sub-Markovian) diffusion semigroups on a Riemannian manifold (possibly with boundary). This inequality as well as the consequent L^2-gradient inequality, are proved to be equivalent to the pointwise curvature lower bound condition together with the convexity or absence of the boundary. Some applications of the log-Harnack inequality are also introduced.

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1. Introduction

Let M be a d-dimensional connected complete Riemannian manifold possibly with a boundary ∂M. Consider $L = \Delta + Z$ for a C^1-vector field Z. Let $X_t(x)$ be the (reflecting) diffusion

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process generated by L with starting point x and life time $\zeta(x)$. Then the associated diffusion semigroup P_t is given by

$$P_t f(x) := \mathbb{E}\big[f\big(X_t(x)\big)1_{\{t<\zeta(x)\}}\big], \quad t \geqslant 0,\ f \in \mathscr{B}_b(M).$$

Although the semigroup depends on Z and the geometry on the whole manifold, we aim to establish Harnack, resp. gradient type inequalities for P_t by using local geometry quantities.

Let $K \in C(M)$ be such that

$$\mathrm{Ric}_Z := \mathrm{Ric} - \nabla Z \geqslant -K, \tag{1.1}$$

i.e. for any $x \in M$ and $X \in T_x M$, $\mathrm{Ric}(X, X) - \langle X, \nabla_X Z\rangle \geqslant -K(x)|X|^2$. Next, for any $D \subset M$, let

$$K(D) := \sup_D K, \qquad D_r = \big\{z \in M:\ \rho(z, D) \leqslant r\big\}, \quad r \geqslant 0,$$

where ρ is the Riemannian distance on M. Finally, to investigate P_t using local curvature bounds, we introduce, for a given bounded open domain $D \subset M$, the following class of reference functions:

$$\mathscr{C}_D = \big\{\phi \in C^2(\bar{D}):\ \phi|_D > 0,\ \phi|_{\partial D \setminus \partial M} = 0,\ N\phi|_{\partial M \cap \partial D} \geqslant 0\big\},$$

where N is the inward unit normal vector field of ∂M. When $\partial M = \emptyset$, the restriction $N\phi|_{\partial M} \geqslant 0$ is automatically dropped. For any $\phi \in \mathscr{C}_D$, we have

$$c_D(\phi) = \sup_D \big\{5|\nabla\phi|^2 - \phi L\phi\big\} \in [0, \infty).$$

The finiteness of $c_D(\phi)$ is trivial since $\bar{D}$ is compact. To see that $c_D(\phi) \geqslant 0$, we consider the following two situations:

(a) There exists $x \in \partial D \setminus \partial M$. We have $\phi(x) = 0$ so that $c_D(\phi) \geqslant \{5|\nabla\phi|^2 - \phi L\phi\}(x) \geqslant 0$.
(b) When $\partial D \setminus \partial M = \emptyset$, we have $\bar{D} = M$. Otherwise, there exists $z \in M \setminus (D \cup \partial M)$. For any $z' \in D \setminus \partial M$, let $\gamma : [0, 1] \to M \setminus \partial M$ be a smooth curve linking z and z'. Since $z' \in D$ but $z \notin D$, there exists $s \in [0, 1]$ such that $\gamma(s) \in \partial D$. This is however impossible since $\partial D \subset \partial M$ and $\gamma(s) \notin \partial M$. Therefore, in this case $M = \bar{D}$ is compact so that the reflecting diffusion process is non-explosive. Now, let $x \in \bar{D}$ such that $\phi(x) = \max_{\bar{D}} \phi$. Since $N\phi|_{\partial M} \geqslant 0$ due to $\phi \in \mathscr{C}_D$, $\phi(X_t) - \phi(x) - \int_0^t L\phi(X_s)\,\mathrm{d}s$ is a sub-martingale so that

$$\phi(x) \geqslant \mathbb{E}\phi(X_t) \geqslant \phi(x) + \int_0^t \mathbb{E}L\phi(X_s)\,\mathrm{d}s, \quad t \geqslant 0.$$

This implies $L\phi(x) \leqslant 0$ (known as the maximum principle) and thus,

$$c_D(\phi) \geqslant \big\{5|\nabla\phi|^2 - \phi L\phi\big\}(x) \geqslant 0.$$

Theorem 1.1. *Let $K \in C(M)$. The following statements are equivalent*:

(1) (1.1) *holds and ∂M is either empty or convex.*
(2) *For any bounded open domain $D \subset M$ and any $\phi \in \mathscr{C}_D$, the log-Harnack inequality*