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Uniqueness results for nonlinear elliptic problems with two lower order terms

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Abstract

In this paper we consider a class of Dirichlet problems for nonlinear elliptic equations of the type

$$\begin{cases} -\operatorname{div}\big(\mathbf{a}(x,\nabla u)\big) - \operatorname{div}\big(\boldsymbol{\Phi}(x,u)\big) + H(x,\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of R^N, $N > 2$, f is an $L^1(\Omega)$ function. We fix some structural conditions on $\mathbf{a}$, $\boldsymbol{\Phi}$ and H to prove uniqueness results for solutions obtained as limit of approximations.

Keywords: Rearrangements; Nonlinear equations; Uniqueness results

1. Introduction

In the present paper we deal with uniqueness for a class of nonlinear elliptic problems of the type

$$\begin{cases} -\operatorname{div}\big(\mathbf{a}(x,\nabla u)\big) - \operatorname{div}\big(\boldsymbol{\Phi}(x,u)\big) + H(x,\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N > 2$ and $2 - \frac{1}{N} < p < N$.

We assume that $\mathbf{a} : (x,z) \in \Omega \times \mathbb{R}^N \to \mathbf{a}(x,z) = (a_i(x,z)) \in \mathbb{R}^N$ is a Carathéodory function satisfying

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$$\mathbf{a}(x,\xi)\cdot\xi \geqslant \lambda|\xi|^p, \quad \xi\in\mathbb{R}^N,\ \lambda>0, \tag{1.2}$$

$$\left|\mathbf{a}(x,\xi)\right| \leqslant \Lambda|\xi|^{p-1}+K, \quad \xi\in\mathbb{R}^N,\ \Lambda, K>0, \tag{1.3}$$

$$\left(\mathbf{a}(x,\xi)-\mathbf{a}(x,\eta)\right)\cdot(\xi-\eta)>0, \quad \xi\neq\eta, \tag{1.4}$$

for almost every $x\in\mathbb{R}^N$ and for every $\xi,\eta\in\mathbb{R}^N$. Moreover $H:(x,\xi)\in\Omega\times\mathbb{R}^N\to H(x,\xi)\in\mathbb{R}$ is a Carathéodory function, differentiable with respect to ξ, satisfying the following conditions

$$\left|\nabla_\xi H(x,\xi)\right| \leqslant b(x)\left(1+|\xi|\right)^{p-2}, \tag{1.5}$$

where $b(x)$ is a nonnegative function belonging to $L^\beta(\Omega)$ with

$$\beta > \frac{2(p-1)N}{2(N-2)-p(N-3)} \quad \text{if } p>2, \tag{1.6}$$

or

$$\beta > \frac{2(p-1)N}{p(N+1)-2N} \quad \text{if } 2-\frac{1}{N}<p\leqslant 2, \tag{1.7}$$

$\boldsymbol{\Phi}:(x,s)\in\Omega\times\mathbb{R}^N\to\boldsymbol{\Phi}(x,s)=(\Phi_i(x,s))\in\mathbb{R}^N$ is a Carathéodory function, differentiable with respect to s, satisfying the following conditions

$$\left|\Phi_s(x,s)\right| \leqslant c(x)\left(1+|s|\right)^{p-2}, \tag{1.8}$$

where $c(x)$ is a nonnegative function belonging to $L^r(\Omega)$ with

$$r > \frac{N(p-1)}{N(2-p)+p(p-1)-1} \quad \text{if } p>2, \tag{1.9}$$

or

$$r > \frac{N(p-1)}{1+N(p-2)} \quad \text{if } 2-\frac{1}{N}<p\leqslant 2. \tag{1.10}$$

Furthermore

$$f\in L^1(\Omega).$$

We recall that when the datum $f\in W^{-1,p'}(\Omega)$ and the coefficients $b(x)$ and $c(x)$ belong respectively to the Lebesgue spaces $L^\beta(\Omega)$ and $L^r(\Omega)$ with $\beta>N$ and $r>\frac{N}{p-1}$, the previous set of hypotheses ensure the existence of a weak solution, i.e. a function $u\in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega \left(\mathbf{a}(x,\nabla u)\cdot\nabla\varphi\right)dx+\int_\Omega \left(\boldsymbol{\Phi}(x,u)\cdot\nabla\varphi\right)dx+\int_\Omega H(x,\nabla u)\varphi\,dx=\int_\Omega f\varphi\,dx, \tag{1.11}$$

for every $\varphi\in W_0^{1,p}(\Omega)$ (see [32] for example). This notion of a solution does not fit if $f\in L^1(\Omega)$.

When $f\in L^1(\Omega)$ and $p=2$, the existence and uniqueness to (1.1) has been proved in [41] by duality arguments under the hypotheses $\|c\|_{L^{\frac{N}{p-1}}}$ or $\|b\|_{L^N}$ is small enough. In particular it has been proved that this solution belongs to $W_0^{1,q}(\Omega)$ for every $q<\frac{N}{N-1}$. Unfortunately such notion of solution cannot be extended to the nonlinear case except to the case $p=2$.

The first attempt to nonlinear case is given in [13,14], where the existence of a distributional solution u to problems of the type (1.1) with $\boldsymbol{\Phi}=0$ and $H=0$ has been proved. Such a solution is