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Reduction of periodic difference systems to linear or autonomous ones ☆

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Abstract

We extend Floquet theory for reducing nonlinear periodic difference systems to autonomous ones (actually linear) by using normal form theory.

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1. Introduction and statement of the main results

The theory of difference equations (or recurrence relations, iterated maps), the methods used in their solutions and their wide applications have advanced beyond their adolescent stage to occupy a central position in applicable analysis. In fact, in the last few years, the proliferation of the subject is witnessed by hundred of research articles and several monographs, see for instance [1,6].

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Among all the attractive topics, the so-called "Floquet theory" is a typical one, which in general deals with periodic linear systems. The asymptotic properties are determined by the periodic map (or monodromy operator) and in particular, by their spectra. So, if concerning periodic orbits of nonlinear systems, the local behavior of a semi-flow close to a periodic orbit is, up to the first order, determined by the derivatives of the flow map on its normal bundle. We refer the works of Coddington and Levinson [4] and Hartman [8] for the classical Floquet theory, to Hale [7] for the case of retarded functional differential equations, to Henry [9] for time periodic linear perturbations of analytic semigroups. Also see [11] for its extension for nonlinear periodic differential equations and to the book [12] for partial differential equations. The motivation of our paper is to extend Floquet theory to reduce the nonlinear periodic difference systems to autonomous ones by studying their normal forms.

In the context of difference equations, we consider linear homogeneous periodic non-autonomous difference systems of the form

$$x_{n+1} = A_n x_n, \tag{1}$$

where $x_n \in \mathbb{R}^d$, $n \in \mathbb{Z}$, A_n is a real $d \times d$ matrix whose entries are function of n satisfying $A_n = A_{n+m}$ for a positive integer m and is *non-degenerated*, i.e., $\det A_n \neq 0$ for all $n \in \mathbb{Z}$. As usual $\mathbb{Z}_+$ denotes the set of non-negative integers and $M = A_{m-1}A_{m-2}\cdots A_0$ denotes the *monodromy* of system (1).

Theorem 1. *There exists a sequence of non-degenerated real matrices $\{B_n\}_{n\in\mathbb{Z}}$, with $B_n = B_{n+m}$ and a constant matrix D such that by the coordinate substitution $x_n = B_n y_n$ system* (1) *is transformed into*

$$y_{n+1} = Dy_n$$

if and only if the monodromy M and the period m satisfy

(1) *m is odd*;
(2) *or M has no negative real eigenvalues*;
(3) *or the Jordan blocks of the Jordan normal form* (*for short, JNF*) *of M corresponding to the negative real eigenvalues appear pairwise.*

This theorem can be seen as an extension of the classical Floquet's theory to periodic difference systems. Detailed discussions can be found in [6,1] and in Section 2 of our paper. We note that the real transformed autonomous system of system (1) can be obtained with at most a $2m$-periodic transformation whatever M and m be.

Consider the m-periodic difference system

$$x_{n+1} = F_n(x_n) = A_n x_n + f_n(x_n), \tag{2}$$

where $x_n \in U$ is a neighborhood of the origin in $\mathbb{R}^d$, A_n is a real $d \times d$ non-degenerated n-depending matrix satisfying $A_n = A_{n+m}$, $f_n = f_{n+m}$ and the coefficients of $f_n : U \to \mathbb{R}^n$ are C^∞ functions such that $f_n(x) = O(\|x\|^2)$ for $n = 0, \ldots, m-1$.

For any matrix A, we denote the set of its eigenvalues by $\lambda(A) = (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d$. For simplicity, sometimes we write $\lambda(A)$ as λ without misunderstanding. The d-tuple $\lambda(A)$ is called *weakly non-resonant* if for $j = 1, \ldots, d$ and $k \in \mathbb{Z}_+^d$, $|k| = \sum_{i=1}^d k_i \geqslant 2$ the following conditions are satisfied

$$\lambda_j \lambda^{-k} \neq e^{\frac{2i\pi}{m}\sqrt{-1}}, \quad i = 1, 2, \ldots, m-1,$$