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On a new embedding theorem and the CLR-type inequality for Euclidean and hyperbolic spaces

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Abstract

The goal of this note is to provide a new embedding theorem and to derive from this embedding the CLR-type inequality for a potential belonging to a proper subspace of integrable functions.

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1. Introduction

We recall that the Sobolev space $H^1(\mathbb{R}^n) := \{D^\alpha u \in L^p(\mathbb{R}^n) \text{ s.t. } |\alpha| \leqslant 1\}$ is not continuously embedded into $L^\infty(\mathbb{R}^n)$, space of bounded functions, when $n \geqslant 2$ [6, Remark 13, p. 284]. To circumvent this impasse, and thanks to the Aharonov–Bohm potential, the authors A.A. Balinsky, W.D. Evans and R.T. Lewis [4] provided a new embedding theorem, for the case $n = 2$, and their proof is based on the results [1,2]. Precisely, they considered the magnetic operator $(-i\nabla + \mathbf{a})^2$, where $\mathbf{a}$ is the Aharonov–Bohm potential, and showed

$$\|u\|_X \leqslant \left[\operatorname{dist}(\widetilde{\phi}, \mathbb{Z})\right]^{-1/2} \left\|(-i\nabla + \mathbf{a})u\right\|_{L^2(\mathbb{R}^2\setminus\{0\})}, \quad \text{for all } u \in H^1_{\mathbf{a}}\left(\mathbb{R}^2 \setminus \{0\}\right),$$

with:

$$\widetilde{\phi} := \tfrac{1}{2\pi}\int_0^{2\pi} \phi(\omega)\, d\omega \notin \mathbb{Z},\ \widetilde{\phi} \text{ stands for the magnetic flux, and } \phi \in L^\infty(\mathbb{S}^1).$$

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$\mathrm{dist}(\widetilde{\phi}, \mathbb{Z}) := \min_{k\in\mathbb{Z}}\{|k - \widetilde{\phi}|\}$.
$X = L^\infty((0,\infty), r\,dr) \otimes L^2(\mathbb{S}^1)$, equipped with the norm

$$\|u\|_X = \operatorname*{ess\,sup}_{r>0}\left(\int_0^{2\pi} \left|u(r,\theta)\right|^2 d\theta\right)^{1/2}.$$

$H^1_{\mathbf{a}}(\mathbb{R}^2 \setminus \{0\})$ is the completion of $C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ w.r.t. the following norm

$$\|u\|_{H^1_{\mathbf{a}}(\mathbb{R}^2\setminus\{0\})} = \left(\int_{\mathbb{R}^2\setminus\{0\}} \left|(-i\nabla + \mathbf{a})u(x)\right|^2 dx + \int_{\mathbb{R}^2\setminus\{0\}} \left|u(x)\right|^2 dx\right)^{1/2}.$$

The expression of $\mathbf{a}$, in polar coordinates, is given by $\mathbf{a}(r,\theta) = \frac{\phi(\theta)}{r}(\operatorname{sen}\theta, -\cos\theta)$.

The scheme of this article, is the following: in Section 2 we state a new embedding theorem for the case $n \geqslant 3$ between the Sobolev space $\widetilde{H^1_0}(\mathbb{R}^n \setminus \overline{B(0,1)})$ where $\overline{B(0,1)}$ is the closure of the unit ball, and the space $\mathscr{X}$, see below for their explicit expressions. Since that there is no CLR inequality—[7,9,18]—in terms of the $L^1(\mathbb{R}^n)$-norm of a potential, hence in Section 3 we derive from our embedding theorem, and by applying the methods from [4], a CLR-type inequality for a Schrödinger operator with a potential belonging to a proper subspace of $L^1(\mathbb{R}^n \setminus \overline{B(0,1)})$. Section 4 is reserved for the study the previous results in a hyperbolic space. We will use several times the closability of a quadratic form [19, §VIII.6] and generalize the techniques issued in [4].

2. An embedding theorem

Let $\mathscr{X}$ be the tensor product space $L^\infty((1,\infty); r^{n-1}\,dr) \otimes \widetilde{L^2}(\mathbb{S}^{n-1}, d\sigma_{\mathbb{S}^{n-1}})$ such that $d\sigma_{\mathbb{S}^{n-1}}$ is the surface element on the unit sphere $\mathbb{S}^{n-1}$, with $\widetilde{L^2}(\mathbb{R}^n \setminus \overline{B(0,1)}) := L^2((1,\infty); r^{n-1}\,dr) \otimes \widetilde{L^2}(\mathbb{S}^{n-1}, d\sigma_{\mathbb{S}^{n-1}})$ and the space $\widetilde{L^2}(\mathbb{S}^{n-1}, d\sigma_{\mathbb{S}^{n-1}})$ is spanned by the orthonormal complete basis of eigenfunctions $(\psi_{k^2})_{k\geqslant 1}$ of $-\Delta_{|_{\mathbb{S}^{n-1}}}$. We recall that the eigenfunctions $(\psi_k)_{k\in\mathbb{N}}$ corresponding to the operator $-\Delta_{|_{\mathbb{S}^{n-1}}}$ constitute an orthonormal basis of the Hilbert space $L^2(\mathbb{S}^{n-1}, d\sigma_{\mathbb{S}^{n-1}})$, i.e., the family $(\psi_k)_{k\in\mathbb{N}}$ satisfies $\int_{\mathbb{S}^{n-1}} \psi_k(\omega).\overline{\psi_l(\omega)}\,d\sigma(\omega) = \delta_{kl}$—the Kronecker delta function. Furthermore, and for each $k \in \mathbb{N}$, $\lambda_k = k(k+n-2)$ is the eigenvalue—with non-trivial multiplicity—associated to ψ_k. We observe that by definition, $\mathscr{X}$ is a subspace of $L^\infty((1,\infty); r^{n-1}\,dr) \otimes L^2(\mathbb{S}^{n-1}, d\sigma_{\mathbb{S}^{n-1}})$, thus $\mathscr{X}$ is endowed with the induced norm and defined by

$$\|u\|_{\mathscr{X}} = \operatorname*{ess\,sup}_{r>1}\left(\left[\int_{\mathbb{S}^{n-1}} \left|u(r,\omega)\right|^2 d\sigma_{\mathbb{S}^{n-1}}(\omega)\right]^{1/2}\right).$$

The quadratic form $\mathfrak{q}(u) = \int_{\mathbb{R}^n\setminus\overline{B(0,1)}} |\nabla u(x)|^2\,dx$ is well defined for $u \in C_0^\infty(\mathbb{R}^n \setminus \overline{B(0,1)})$—space of smooth functions with compact support in $\mathbb{R}^n \setminus \overline{B(0,1)}$. Plus, $\mathfrak{q}$ is closable, i.e., $C_0^\infty(\mathbb{R}^n \setminus \overline{B(0,1)})$ is complete w.r.t. the norm

$$\|u\|_{H^1_0(\mathbb{R}^n\setminus\overline{B(0,1)})} = \left(\int_{\mathbb{R}^n\setminus\overline{B(0,1)}} \left|\nabla u(x)\right|^2 dx + \int_{\mathbb{R}^n\setminus\overline{B(0,1)}} \left|u(x)\right|^2 dx\right)^{1/2}.$$

$H^1_0(\mathbb{R}^n \setminus \overline{B(0,1)})$ is the completion of $C_0^\infty(\mathbb{R}^n \setminus \overline{B(0,1)})$. Therefore, $\mathfrak{q}$ is associated to a unique self-adjoint operator, namely the operator $-\Delta$. Now, we are able to state our embedding theorem.