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Computing residue currents of monomial ideals using comparison formulas

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Abstract

Given a free resolution of an ideal $\mathfrak{a}$ of holomorphic functions, one can construct a vector-valued residue current R, which coincides with the classical Coleff–Herrera product if $\mathfrak{a}$ is a complete intersection ideal and whose annihilator ideal is precisely $\mathfrak{a}$.

We give a complete description of R in the case when $\mathfrak{a}$ is an Artinian monomial ideal and the resolution is the hull resolution (or a more general cellular resolution). The main ingredient in the proof is a comparison formula for residue currents due to the first author.

By means of this description, we obtain in the monomial case a current version of a factorization of the fundamental cycle of $\mathfrak{a}$ due to Lejeune-Jalabert.

MSC: 32A27; 13D02

1. Introduction

With a regular sequence $f_1, \dots, f_p$ of holomorphic functions at the origin in $\mathbf{C}^n$, there is a canonical associated residue current, the *Coleff–Herrera product* $R^f_{CH} = \bar\partial[1/f_p] \wedge \cdots \wedge \bar\partial[1/f_1]$, introduced in [10]. It has support on $\{f_1 = \cdots = f_p = 0\}$ and satisfies the *duality principle* [11,20]: *A holomorphic function ξ is locally in the ideal (f) generated by $f_1, \dots, f_p$ if and only if ξ annihilates R^f_{CH}, i.e., $\xi R^f_{CH} = 0$.* Given a free resolution of an ideal (sheaf) $\mathfrak{a}$ of holomorphic functions, Andersson and the second author constructed in [5] a vector-valued residue current

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R that satisfies the duality principle and that coincides with R^f_{CH} if $\mathfrak{a}$ is a complete intersection ideal, generated by a regular sequence $f_1, \dots, f_p$, see Section 2. This construction has recently been used, e.g., to obtain new results for the $\bar\partial$-equation and effective solutions to polynomial ideal membership problems on singular varieties, see, e.g., [2–4,7,24].

In this paper we compute the current R for the *hull resolution* (and more general cellular resolutions), introduced by Bayer and Sturmfels [8], of Artinian, i.e., 0-dimensional, monomial ideals, extending previous results by the second author. The hull resolution of a monomial ideal M is encoded in the *hull complex* $\operatorname{hull}(M)$, which is a labeled polyhedral cell complex in $\mathbf{R}^n$ of dimension $n-1$ with one vertex for each minimal generator of M. The face $\sigma \in \operatorname{hull}(M)$ is labeled by the least common multiple of the monomials corresponding to the vertices of σ, see Section 4.

Theorem 1.1. *Let M be an Artinian monomial ideal in $\mathbf{C}^n$ and let R be the residue current constructed from the hull resolution of M. Then R has one entry R_σ for each $(n-1)$-dimensional face σ of* $\operatorname{hull}(M)$*, and*

$$R_\sigma = \bar\partial\left[\frac{1}{z_n^{\alpha_n}}\right] \wedge \cdots \wedge \bar\partial\left[\frac{1}{z_1^{\alpha_1}}\right],$$

where $z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ is the label of σ.

If M is a complete intersection ideal, $\operatorname{hull}(M)$ is an $(n-1)$-simplex and the hull resolution is the Koszul complex. In general, $\operatorname{hull}(M)$ is a polyhedral subdivision of an $(n-1)$-simplex. In fact, Theorem 1.1 holds for more general cellular resolutions, where the underlying polyhedral cell complex is a polyhedral subdivision of the $(n-1)$-simplex, see Theorem 5.1.

It was proved in [10] that if $f_1, \dots, f_p$ is a regular sequence, then

$$R^f_{CH} \wedge \frac{df_1 \wedge \cdots \wedge df_p}{(2\pi i)^p} = \big[(f)\big], \tag{1.1}$$

where $[(f)]$ is the fundamental cycle of the ideal (f). Our main motivation to compute R explicitly was to understand a similar factorization of the fundamental cycle of an arbitrary ideal. By computing $d\varphi := d\varphi_0 \circ \cdots \circ d\varphi_{n-1}$, where φ_k are the maps in the (hull) resolution of a (generic) Artinian monomial ideal $\mathfrak{a}$, and using Theorem 1.1, we get

$$\frac{d\varphi}{n!(2\pi i)^n} \circ R = [\mathfrak{a}], \tag{1.2}$$

see Section 7. Since $\mathfrak{a}$ is Artinian, $[\mathfrak{a}] = m[0]$, where m is the *geometric multiplicity* $\dim_{\mathbf{C}} \mathcal{O}_0^n/\mathfrak{a}$ of $\mathfrak{a}$, see [14, Section 1.5]. Moreover, since $\mathfrak{a}$ is monomial, m equals the volume of the *staircase* $\mathbf{R}^n_+ \setminus \bigcup_{z^\alpha \in \mathfrak{a}} \{\alpha + \mathbf{R}^n_+\}$ of $\mathfrak{a}$. If $\mathfrak{a}$ is a complete intersection ideal generated by $f_1, \dots, f_n$, then $d\varphi = n!\, df_1 \wedge \cdots \wedge df_n$, and thus (1.2) can be seen as a generalization of (1.1). We recently managed to prove a generalized version of (1.2) for arbitrary ideals of pure dimension; this is a current version of (a generalization of) a result due to Lejeune-Jalabert [17] and will be the subject of the forthcoming paper [16].

In [27] the current R was computed as the push-forward of a certain current in a toric resolution of the ideal M. The main result in that paper asserts that each R_σ is of the form $R_\sigma = c_\sigma \bar\partial[1/z_n^{\alpha_n}] \wedge \cdots \wedge \bar\partial[1/z_1^{\alpha_1}]$ for some $c_\sigma \in \mathbf{C}$. The coefficients c_σ appear as integrals that seem to be hard to compute in general, see Section 6. The proof of Theorem 1.1 given here is