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Isochronicity conditions for some planar polynomial systems II

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Abstract

We study the isochronicity of centers at $O \in \mathbb{R}^2$ for systems

$$\dot{x} = -y + A(x, y), \qquad \dot{y} = x + B(x, y),$$

where $A, B \in \mathbb{R}[x, y]$, which can be reduced to the Liénard type equation. When $\deg(A) \leqslant 4$ and $\deg(B) \leqslant 4$, using the so-called C-algorithm we found 36 new multiparameter families of isochronous centers. For a large class of isochronous centers we provide an explicit general formula for linearization. This paper is a direct continuation of a previous one with the same title [Islam Boussaada, A. Raouf Chouikha, Jean-Marie Strelcyn, Isochronicity conditions for some planar polynomial systems, Bull. Sci. Math. 135 (1) (2011) 89–112], but it can be read independently.

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1. Introduction

Let us consider the system of real differential equations of the form

$$\frac{dx}{dt} = \dot{x} = -y + A(x, y), \qquad \frac{dy}{dt} = \dot{y} = x + B(x, y), \tag{1.1}$$

where (x, y) belongs to an open connected subset $U \subset \mathbb{R}^2$ containing the origin $O = (0, 0)$, with $A, B \in C^1(U, \mathbb{R})$ such that A and B as well as their first partial derivatives vanish at O. An isolated singular point $p \in U$ of system (1.1) is a *center* if there exists a punctured neighborhood $V \subset U$ of p such that every orbit of (1.1) lying in V is a closed orbit surrounding p. A center p is *isochronous* if the period is constant for all closed orbits in some neighborhood of p.

The simplest example is the linear system with an isochronous center at the origin O:

$$\dot{x} = -y, \qquad \dot{y} = x. \tag{1.2}$$

The problem of characterization of couples (A, B) such that O is an isochronous center (even for a center) for the system (1.1) is largely open.

The well-known Poincaré theorem asserts that when A and B are real analytic, a center of (1.1) at the origin O is isochronous if and only if in some real analytic coordinate system it takes the form of the linear center (1.2) (see for example [1, Theorem 13.1], and [20, Theorem 4.2.1]).

An overview [6] presents the basic results concerning the problem of the isochronicity, see also [1,10–12,20]. As this paper is a direct continuation of [3], we refer the reader to it for general introduction to the subject. Here we will recall only the strictly necessary facts.

In some circumstances system (1.1) can be reduced to *the Liénard type equation*

$$\ddot{x} + f(x)\dot{x}^2 + g(x) = 0 \tag{1.3}$$

with $f, g \in C^1(J, \mathbb{R})$, where J is some neighborhood of $0 \in \mathbb{R}$ and $g(0) = 0$. In this case, system (1.1) is called *reducible*. Eq. (1.3) is associated to the equivalent, two-dimensional, Liénard type system

$$\left.\begin{aligned} &\dot{x} = y, \\ &\dot{y} = -g(x) - f(x)y^2. \end{aligned}\right\} \tag{1.4}$$

For reducible systems considered in this paper, the nature (center and isochronicity) of the singular point O for both systems (1.1) and (1.4) is the same.

Let us return now to the Liénard type equation (1.3). Let us define the following functions

$$F(x) := \int_0^x f(s)\,ds, \qquad \phi(x) := \int_0^x e^{F(s)}\,ds. \tag{1.5}$$

The first integral of the system (1.4) is given by the formula [21, Theorem 1]

$$I(x, \dot{x}) = \frac{1}{2}\bigl(\dot{x}e^{F(x)}\bigr)^2 + \int_0^x g(s)e^{2F(s)}\,ds. \tag{1.6}$$

When $xg(x) > 0$ for $x \neq 0$, define the function X by

$$\frac{1}{2}\xi(x)^2 = \int_0^x g(s)e^{2F(s)}\,ds \tag{1.7}$$

and $x\xi(x) > 0$ for $x \neq 0$.