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Maximal eigenvalue and norm of a product of Toeplitz matrices. Study of a particular case

Valeur propre maximale et norme d'un produit de matrices de Toeplitz. Etude d'un cas particulier

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Abstract

In this paper we describe the asymptotic behaviour of the spectral norm of the product of two finite Toeplitz matrices as the matrix dimension goes to infinity. These Toeplitz matrices are generated by functions with Fisher–Hartwig singularities of negative order. If these functions are positives the product of the two matrices has positive eigenvalues and it is known that the spectral norm is also the largest eigenvalue of this product.

Résumé

Dans cet article nous donnons une expression asymptotique de la norme d'un produit de deux matrices de Toeplitz dont les symboles admettent une ou plusieurs singularités de Fisher–Hartwig d'ordre négatif. Si ces symboles sont strictement positifs les valeurs propres du produit de matrices étudié sont également strictement positives et la norme spectrale est alors égale à la valeur propre maximale.

MSC: primary 47B39, 47B35; secondary 47BXX

Keywords: Spectral norm of a product of Toeplitz matrices; Maximal eigenvalue of a product of Toeplitz matrices; Long range memory; Fisher–Hartwig singularities of negative order

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Mots-clés : Norme spectrale d'un produit de matrices de Toeplitz ; Valeur propre maximale d'un produit de matrices de Toeplitz ; Longue mémoire ; Singularités de Fisher–Hartwig d'ordre négatif

1. Introduction

If $f \in L^1(\mathbb{T})$ the Toeplitz matrix with symbol f denoted by $T_N(f)$ is the $(N+1)\times(N+1)$ matrix such that

$$\big(T_N(f)\big)_{i+1,j+1} = \hat{f}(j-i), \quad \forall i,j,\ 0 \leqslant i,j \leqslant N$$

(see, for instance, [6,7]). We say that a function h is regular if $h \in L^\infty(\mathbb{T})$ and $h > 0$. Otherwise the function h is said singular. If b is a function in $L^\infty(\mathbb{T})$ that is continuous in $z_r \in \mathbb{T}$ we call Fisher–Hartwig symbols the functions

$$f(z) = b(z)\prod_{r=1}^{R}|z-z_r|^{2\alpha_r}\varphi_{\beta_r,z_r}(z) \quad (z \in \mathbb{T})$$

where

- the complex numbers α_r and β_r are subject to the constraints $-\frac{1}{2} < \alpha_r < \frac{1}{2}$ and $-\frac{1}{2} < \beta_r < \frac{1}{2}$,
- the functions φ_{β_r,z_r} are defined as $\varphi_{\beta_r,z_r}(z) = e^{i\beta_r \arg(-\frac{z}{z_r})}$, with $\arg(z) \in (-\pi,\pi]$.

The problem of the extreme eigenvalues of a Toeplitz matrix is well known (see [12] and [1]). If $\lambda_{k,N}$, $1 \leqslant k \leqslant N+1$ are the eigenvalues of $T_N(f)$ with $f \geqslant 0$ we have $0 \leqslant m_f \leqslant \lambda_{1,N} \leqslant \lambda_{2,N} \leqslant \cdots \leqslant \lambda_{N+1,N} \leqslant M_f$ and

$$\lim_{N\to=+\infty}\lambda_{1,N} = m_f \quad \text{and} \quad \lim_{N\to+\infty}\lambda_{N+1,N} = M_f$$

with $m_f = \operatorname{essinf} f$ and $M_f = \operatorname{essup} f$. In [5] and [8] Böttcher and Grudsky on one hand and Böttcher and Virtanen in the other hand give an asymptotic estimation of the maximal eigenvalue in the case of one Toeplitz matrix when the symbol has one or several zeros of negative order. In [13] we have obtained the asymptotic of the minimal eigenvalue of one Toeplitz matrix when the symbol has one zero of order α with $\alpha > \frac{1}{2}$.

But estimating the eigenvalues of the product of two Toeplitz matrices is more delicate. Effectively it is clear that a product of Toeplitz matrices is generally not a Toeplitz matrix. In the first part of this paper we consider the product $T_N(f_1)T_N(f_2)$ of two Toeplitz matrices where $f_1(e^{i\theta}) = |e^{i\theta} - e^{i\theta_0}|^{-2\alpha_1}c_1(e^{i\theta})$, and $f_2(e^{i\theta}) = |e^{i\theta} - e^{i\theta_0}|^{-2\alpha_2}c_2(e^{i\theta})$ with $0 < \alpha_1, \alpha_2 < \frac{1}{2}$ and $c_1, c_2 \in L^\infty(\mathbb{T})$ that are continuous and nonzero in $e^{i\theta_0}$. For these symbols we obtain the norm of the matrix $T_N(f_1)T_N(f_2)$. Owing to an important result of Widom (see Lemma 1 and also [17,16,15,9]), which connects the norm of an operator and the norm of a matrix. A proof of this last result can be found in [8]. If we assume that c_1, c_2 are two regular functions, the matrix $T_N(f_1)T_N(f_2)$ has positive eigenvalues and its norm is also the maximal eigenvalue of this matrix. Hence our main result (see Theorem 3) can be also stated as

Theorem 1. *Let $f_1(e^{i\theta}) = |e^{i\theta_0} - e^{i\theta}|^{-2\alpha_1}c_1(e^{i\theta})$ and $f_2(e^{i\theta}) = |e^{i\theta_0} - e^{i\theta}|^{-2\alpha_2}c_2(e^{i\theta})$ with $0 < \alpha_1, \alpha_2 < \frac{1}{2}$ and c_1, c_2 be two regular functions continuous at $z_0 = e^{i\theta_0}$. Then if $\Lambda_{\alpha_1,\alpha_2,N}$ is the maximal eigenvalue of $T_N(f_1)T_N(f_2)$ we have*