Bull. Sci. math. 137 (2013) 746–774

BULLETIN DES SCIENCES MATHÉMATIQUES

www.elsevier.com/locate/bulsci

On the number of invariant conics for the polynomial vector fields defined on quadrics

Yudy Bolaños [a,*], Jaume Llibre [a], Claudia Valls [b]

[a] *Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain*
[b] *Departamento de Matemática, Instituto Superior Técnico, Universidade Técnica de Lisboa, Av. Rovisco Pais 1049-001, Lisboa, Portugal*

Received 3 March 2013

Available online 12 April 2013

Abstract

The quadrics here considered are the nine real quadrics: parabolic cylinder, elliptic cylinder, hyperbolic cylinder, cone, hyperboloid of one sheet, hyperbolic paraboloid, elliptic paraboloid, ellipsoid and hyperboloid of two sheets. Let $\mathcal{Q}$ be one of these quadrics. We consider a polynomial vector field $\mathcal{X} = (P, Q, R)$ in $\mathbb{R}^3$ whose flow leaves $\mathcal{Q}$ invariant. If $m_1 =$ degree P, $m_2 =$ degree Q and $m_3 =$ degree R, we say that $\mathbf{m} = (m_1, m_2, m_3)$ is the degree of $\mathcal{X}$. In function of these degrees we find a bound for the maximum number of invariant conics of $\mathcal{X}$ that results from the intersection of invariant planes of $\mathcal{X}$ with $\mathcal{Q}$. The conics obtained can be degenerate or not. Since the first six quadrics mentioned are ruled surfaces, the degenerate conics obtained are formed by a point, a double straight line, two parallel straight lines, or two intersecting straight lines; thus for the vector fields defined on these quadrics we get a bound for the maximum number of invariant straight lines contained in invariant planes of $\mathcal{X}$. In the same way, if the conic is non-degenerate, it can be a parabola, an ellipse or a hyperbola and we provide a bound for the maximum number of invariant non-degenerate conics of the vector field $\mathcal{X}$ depending on each quadric $\mathcal{Q}$ and of the degrees m_1, m_2 and m_3 of $\mathcal{X}$.

MSC: 34A34; 34C05; 34C14

Keywords: Polynomial vector fields; Invariant quadrics; Invariant conics; Extactic polynomial

* Corresponding author.
E-mail addresses: ymbolanos@gmail.com (Y. Bolaños), jllibre@mat.uab.cat (J. Llibre), cvalls@math.ist.utl.pt (C. Valls).

0007-4497/$ – see front matter
http://dx.doi.org/10.1016/j.bulsci.2013.04.003

1. Introduction

In this paper we deal with polynomial vector fields

$$\mathcal{X} = P(x, y, z)\frac{\partial}{\partial x} + Q(x, y, z)\frac{\partial}{\partial y} + R(x, y, z)\frac{\partial}{\partial z},$$

where P, Q and R are polynomials of degrees m_1, m_2 and m_3 respectively in the variables x, y, z with coefficients in $\mathbb{R}$. We say that $\mathbf{m} = (m_1, m_2, m_3)$ is the *degree* of the polynomial vector field. We assume that $\mathcal{X}$ has an invariant quadric $\mathcal{Q}$, then we say that $\mathcal{X}$ is a polynomial vector field defined on the quadric $\mathcal{Q}$.

An invariant plane of $\mathcal{X}$ intersects the invariant quadric $\mathcal{Q}$ in an invariant conic. Our main objective is to study the maximum number of invariant conics of this kind that the polynomial vector field $\mathcal{X}$ can have in function of its degrees m_1, m_2 and m_3. If the conic is non-degenerate, then it is an ellipse, a parabola or a hyperbola. If it is degenerate, then it is formed by a point, two straight lines that intersect at a point, or two parallel straight lines, or a double straight line.

The study of the maximum number of invariant classes of algebraic curves in $\mathbb{R}^2$, of invariant classes of surfaces in $\mathbb{R}^3$, and of invariant classes of hypersurfaces in $\mathbb{R}^n$ have been studied recently for several authors. Thus, the maximum number of straight lines that a polynomial vector field in $\mathbb{R}^2$ can have in function of its degree has been studied in [1,14,17]. The maximum number of algebraic limit cycles that a polynomial vector field in $\mathbb{R}^2$ can have has been studied in [9,10,18]. The maximum number of invariant meridians and parallels for polynomial vector fields on a 2-dimensional torus have been considered in [8,11]. The maximum number of invariant hyperplanes (respectively $\mathbb{S}^{n-1}$ spheres) that polynomial vector fields can have in $\mathbb{R}^n$ have been determined in [7] (respectively [2]).

The paper is organized as follows. In Section 2 we provide some basic definitions that will need later on. Next we introduce the quadrics and theirs canonical forms, and in the following sections we give the bounds for the maximum number of the invariant conics living on invariant planes that a polynomial vector field defined on a quadric can have.

2. Basic definitions and results

We recall that the *polynomial differential system* in $\mathbb{R}^3$ of degree $\mathbf{m}$ associated with the vector field $\mathcal{X}$ is

$$\frac{dx}{dt} = P(x, y, z), \qquad \frac{dy}{dt} = Q(x, y, z), \qquad \frac{dz}{dt} = R(x, y, z).$$

Let $\mathcal{Q}$ be a quadric given by $\mathcal{Q} = \{(x, y, z) \in \mathbb{R}^3\colon\ G(x, y, z) = 0\}$, where $G : \mathbb{R}^3 \to \mathbb{R}$ is a polynomial of degree 2. We say that $\mathcal{X}$ defines a *polynomial vector field on the quadric* $\mathcal{Q}$ if $\mathcal{X}G = (P, Q, R) \cdot \nabla G = 0$ on all points of $\mathcal{Q}$. Here, as usual ∇G denotes the gradient of the function G.

We denote by $\mathbb{R}[x, y, z]$ the ring of all polynomials in the variables x, y, z with coefficients in $\mathbb{R}$. Let $f(x, y, z) \in \mathbb{R}[x, y, z] \setminus \mathbb{R}$. We say that $\{f = 0\} \cap \mathcal{Q} \subset \mathbb{R}^3$ is an *invariant algebraic curve of the vector field* $\mathcal{X}$ *on* $\mathcal{Q}$ (or simply an *invariant algebraic curve of* $\mathcal{Q}$) if it satisfies:

(i) There exists a polynomial $k \in \mathbb{R}[x, y, z]$ such that

$$\mathcal{X} f = P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} + R\frac{\partial f}{\partial z} = kf \quad \text{on } \mathcal{Q},$$

the polynomial k is the *cofactor* of $f = 0$ on $\mathcal{Q}$; and