

On the first integrals in the center problem

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Abstract

In this paper we deal with the analyticity of the first integral of any (possibly degenerate) center of an analytic planar differential system.

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1. Introduction

One of the more classical problems in the qualitative theory of planar real analytic differential systems is the distinction between a center and a focus, the so-called *center–focus problem* and its connection with the integrability problem, see for instance [3,5]. Let $p \in \mathbb{R}^2$ be a singular point of a differential system in $\mathbb{R}^2$. We say that p is a *center* if there is a neighborhood U of p such that all the orbits of $U \setminus \{p\}$ are periodic, and we say that p is a *focus* if there is a neighborhood U of p such that all the orbits of $U \setminus \{p\}$ spiral either in forward or in backward time to p . Once we have a center at p of a differential system in $\mathbb{R}^2$, another problem is to know if there exists or not a first integral H defined in some neighborhood U of p (i.e. a non-constant function $H : U \rightarrow \mathbb{R}$ such that H is constant on the orbits of the differential system), and to know the differentiability of H .

More specifically, we consider the real analytic system of differential equations given by

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \quad (1)$$

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where $\mathcal{X} := P(x, y)\partial_x + Q(x, y)\partial_y$ is a planar vector field defined in a domain $U \subset \mathbb{R}^2$. We shall always assume that $0 \in U$ is an isolated singularity of $\mathcal{X}$, i.e. $\mathcal{X}(0) = 0$, which is a center. A center is called linear type (also non-degenerate) or nilpotent if, after an affine change of variables and a rescaling of the time variable, the linear part of $\mathcal{X}$ at 0 takes the form $-y\partial_x + x\partial_y$ or $y\partial_x$ respectively. If the linear part of $\mathcal{X}$ at 0 is identically zero then the center is called degenerate. For linear type and nilpotent singular points the center–focus problem is solved in some sense (see [19,13,20,16,1] for a precise formulation), but for degenerate centers it is still wide open, see e.g. [8,11] and references therein.

The problem of characterizing center singularities is intimately related to the integrability of $\mathcal{X}$, i.e., to the existence of a first integral in a neighborhood of the singularity. The well-known Lyapunov–Poincaré theorem [19,13,17] ensures that 0 is a linear center if and only if $\mathcal{X}$ admits a C^ω first integral in a neighborhood of the origin. If 0 is a nilpotent or degenerate center, then $\mathcal{X}$ does not need to admit an analytic first integral, as illustrated with several examples in [18,17,4]. However, if we allow for lower regularity of the first integral, Mazzi and Sabatini proved [15] that any (possibly degenerate) center has a C^∞ first integral and also the existence of a C^∞ integrating factor in a neighborhood of the singularity. In [12] it is proved that any (possibly degenerate) center admits a C^∞ inverse integrating factor and a C^∞ Lie symmetry in a neighborhood of the singularity. Moreover in [12] it was shown that there exist analytic degenerate centers for which any C^∞ inverse integrating factor is flat at the singular point, and that a sufficient condition for the existence of an analytic inverse integrating factor is the existence of an analytic first integral.

In this paper we study the problem of determining explicitly the first integral of a nilpotent or degenerate center in the case that this first integral is not analytic in a neighborhood of the singularity. For non-degenerate centers we also study the case when a C^∞ first integral is known but due to the Lyapunov–Poincaré theorem the system must have a C^ω one.

2. Main results about centers

The following theorem given in [10] characterizes any center of an analytic differential system in $\mathbb{R}^2$ and it is a straightforward corollary of the results given in [15].

Theorem 1 (*Center Theorem*). *A singular point p of a real analytic system is a center if and only if p is monodromic and there exists a C^∞ first integral defined in some neighborhood of p .*

The characterization of the linear type centers in terms of the existence of an analytic first integral due to Poincaré [19] and Lyapunov [13] can be written as:

Theorem 2 (*Linear Type Center Theorem*). *The real analytic system*

$$\dot{x} = -y + F_1(x, y), \quad \dot{y} = x + F_2(x, y),$$

where F_1 and F_2 start with terms of order higher than 1, has a center at the origin if and only if there exists a local analytic first integral of the form $H = x^2 + y^2 + F(x, y)$ defined in a neighborhood of the origin, where F starts with terms of order higher than 2.

In [8] the centers of planar analytic vector fields which are limit of linear type centers are studied. It is proved that all the nilpotent centers are limit of linear type centers and consequently the Poincaré–Lyapunov method to find linear type centers can also be used to find all the nilpotent

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