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# Admissibility and nonuniform exponential dichotomy on the half-line

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**Abstract**

The aim of this paper is to deduce new conditions for the existence of the nonuniform exponential dichotomy of evolution families on the half-line. We consider an evolution family having a nonuniform exponential growth and we associate to it an input–output equation. We prove that the admissibility of the pair $(C_b(\mathbb{R}_+, X), L^p(\mathbb{R}_+, X))$ with respect to this equation implies the existence of a nonuniform exponential dichotomy. We also present an illustrative example which shows that, generally, the converse implication is not valid in the nonuniform case. Finally, we give an application to the case of uniform exponential dichotomy.



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*Keywords:* Nonuniform exponential dichotomy; Evolution family; Admissibility

## 1. Introduction

One of the most interesting problems in the asymptotic theory of dynamical systems is to determine conditions for the splitting of the state space into a direct sum of invariant subspaces on which the behavior is modeled by exponential decay backward and forward in time, this phenomenon being described by exponential dichotomy. In the last decades, the existence and

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robustness of the dichotomy were investigated developing a large variety of methods (see [2,3,5,7,9,11–15,17–26]), the dichotomy being an important tool in the study of the asymptotic behavior of nonlinear systems (see [2,4,27] and the references therein). Among the various approaches used in order to detect the existence of an exponential dichotomy, a special class of methods is represented by the so-called admissibility techniques or input–output methods (see [3–5,7–9,11–15,20–25]). These techniques have a long and impressive history that goes back to the works of Perron (see [16]), Massera and Schäffer (see [10]), Daleckii and Krein (see [8]) and Coppel (see [6]). In the classical works, the main idea was to establish conditions for stability and dichotomy of the nonautonomous system

$$\dot{x}(t) = A(t)x(t), \quad t \in J, \ J \in \{\mathbb{R}_+, \mathbb{R}\}$$

in a Banach space $X$, in terms of some properties of the operator

$$Px(t) = \dot{x}(t) - A(t)x(t), \quad t \in J$$

in a space of $X$-valued functions. In recent years, the case of nonautonomous systems was treated in the unified setting of evolution families $\mathcal{U} = \{U(t,s)\}_{t \geqslant s, t,s \in J}$, with $J \in \{\mathbb{R}_+, \mathbb{R}\}$, and instead of the operator $P$, one started to investigate the associated integral equation

$$f(t) = U(t,s)f(s) + \int_s^t U(t,\tau)v(\tau)\,d\tau, \quad t \geqslant s, \ t,s \in J, \tag{$E_{\mathcal{U}}$}$$

in various function spaces (see [9,11,12,14,15,20–25]). A remarkable step in this framework was made by Van Minh, Räbiger and Schnaubelt in [14], where the authors obtained the following characterization for uniform exponential dichotomy on the half-line:

**Theorem 1.1.** *Let $\mathcal{U} = \{U(t,s)\}_{t \geqslant s \geqslant 0}$ be an evolution family on a Banach space $X$. Let $C_0(\mathbb{R}_+, X) = \{u : \mathbb{R}_+ \to X$ continuous: $u(0) = \lim_{t\to\infty} u(t) = 0\}$. Then the following assertions are equivalent*:

(i) *$\mathcal{U}$ has a uniform exponential dichotomy*;
(ii) *for every $v \in C_0(\mathbb{R}_+, X)$ Eq. $(E_{\mathcal{U}})$ has a solution $f \in C_0(\mathbb{R}_+, X)$ and the subspace $\tilde{X}_0 := \{x \in X:\ U(\cdot,0)x \in C_0(\mathbb{R}_+, X)\}$ is closed and complemented in $X$.*

This result was the starting point for a number of studies devoted to the existence of the uniform exponential dichotomy on the half-line, the techniques being developed and improved for various classes of input and output spaces (see [9,12,15,22,23,25]). Despite the explosive development in this framework, it is worth mentioning that all these studies were devoted to the case of evolution families with *uniform* exponential growth, i.e. there are $M, \omega > 0$ such that

$$\|U(t,s)\| \leqslant Me^{\omega(t-s)}, \quad \forall t,s \in J, \ t \geqslant s. \tag{1.1}$$

Although there are substantial differences between the techniques used in the study of the dichotomy on the whole line ($J = \mathbb{R}$) compared with those considered in order to detect the existence of the dichotomy on the half-line ($J = \mathbb{R}_+$), it should be noted that the integral admissibility methods represented, in the same time, important tools in the investigation of the dichotomy on the whole line, but the attention was still focused on the case of evolution families with *uniform* exponential growth (see [20,21,24] and the references therein).