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A Bernstein-type inequality for rational functions in weighted Bergman spaces

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Abstract

Given $n \geqslant 1$ and $r \in (0, 1)$, we consider the set $\mathcal{R}_{n,r}$ of rational functions of degree at most n with no poles in $\frac{1}{r}\mathbb{D}$, where $\mathbb{D}$ is the unit disc of the complex plane. We give an asymptotically sharp Bernstein-type inequality for functions in $\mathcal{R}_{n,r}$ in weighted Bergman spaces with "sub-polynomially" decreasing weights. We also prove that this result cannot be extended to weighted Bergman spaces with "super-polynomially" decreasing weights.

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1. Introduction

Estimates of the norms of derivatives for polynomials and rational functions (in different functional spaces) is a classical topic of complex analysis (see surveys by A.A. Gonchar [10], V.N. Rusak [16], and P. Borwein and T. Erdélyi [3, Chapter 7]). Such inequalities have applications in many domains of analysis; to mention just some of them: (1) matrix analysis and

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operator theory (e.g., "Kreiss Matrix Theorem" [12,17] or resolvent estimates of power bounded matrices [19,18]), (2) inverse theorems of rational approximation (see [4,15,14]), (3) effective Nevanlinna–Pick interpolation problems (see [22,23]).

Here we present Bernstein-type inequalities for rational functions f of degree n with poles in $\{z\colon |z|>1\}$, involving Hardy norms and weighted Bergman norms. Let $\mathcal{P}_n$ be the complex space of polynomials of degree at most $n\geqslant 1$. Let $\mathbb{D}=\{z\in\mathbb{C}\colon |z|<1\}$ be the unit disc of the complex plane and $\overline{\mathbb{D}}=\{z\in\mathbb{C}\colon |z|\leqslant 1\}$ its closure. Given $r\in(0,1)$, we define

$$\mathcal{R}_{n,r}=\left\{\frac{p}{q}\colon\ p,q\in\mathcal{P}_n,\ q(\xi)\neq 0 \text{ for } |\xi|<\frac{1}{r}\right\},$$

the set of all rational functions of degree at most $n\geqslant 1$, with no poles in $\frac{1}{r}\mathbb{D}$. We extend the definition of $\mathcal{R}_{n,r}$ to the case $r=0$ by assuming $\mathcal{R}_{n,0}=\mathcal{P}_n$.

1.1. Definitions of Hardy spaces and radial weighted Bergman spaces

We denote by $\mathrm{Hol}(\mathbb{D})$ the space of all holomorphic functions on $\mathbb{D}$. From now on, if $f\in\mathrm{Hol}(\mathbb{D})$ then for every $\rho\in(0,1)$ we define

$$f_\rho:\xi\mapsto f(\rho\xi),\quad \xi\in\frac{1}{\rho}\mathbb{D}.$$

We consider the following two scales of Banach spaces $X\subset\mathrm{Hol}(\mathbb{D})$:

(a) The Hardy spaces $H^p=H^p(\mathbb{D})$, $1\leqslant p\leqslant\infty$:

$$H^p=\left\{f\in\mathrm{Hol}(\mathbb{D})\colon\ \|f\|_{H^p}^p=\sup_{0\leqslant\rho<1}\int_{\mathbb{T}}\left|f_\rho(\xi)\right|^p\,\mathrm{d}m(\xi)<\infty\right\},$$

where m stands for the normalized Lebesgue measure on $\mathbb{T}=\{z\in\mathbb{C}\colon |z|=1\}$. As usual, we denote by H^∞ the space of all bounded analytic functions in $\mathbb{D}$.

(b) The radial weighted Bergman spaces $L_a^p(w)$, $1\leqslant p<\infty$ (where "a" means analytic):

$$L_a^p(w)=\left\{f\in\mathrm{Hol}(\mathbb{D})\colon\ \|f\|_{L_a^p(w)}^p=\int_0^1\rho w(\rho)\int_{\mathbb{T}}\left|f_\rho(\xi)\right|^p\,\mathrm{d}m(\xi)\,\mathrm{d}\rho<\infty\right\},$$

where the weight w satisfies $w\geqslant 0$ and $\int_0^1 w(\rho)\,\mathrm{d}\rho<\infty$. For the classical power weights $w(\rho)=w_\beta(\rho)=(1-\rho)^\beta$, $\beta>-1$, we have $L_a^p(w_\beta)=L_a^p((1-|z|)^\beta\,\mathrm{d}A(z))$, A being the normalized area measure on $\mathbb{D}$.

For general properties of these spaces we refer to [11,24].

From now on, for two positive functions a and b, we say that a is dominated by b, denoted by $a\lesssim b$, if there is a constant $c>0$ such that $a\leqslant cb$; and we say that a and b are comparable, denoted by $a\asymp b$, if both $a\lesssim b$ and $b\lesssim a$.

1.2. Statement of the problem and known results

By Bernstein-type inequalities for rational functions one usually understands the inequalities of the form