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Operators on a nonlocally compact group algebra

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Abstract

The article is devoted to the investigation of operators on a nonlocally compact group algebra. Their isomorphisms are also studied.

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1. Introduction

Group algebras play very important role in algebra, harmonic analysis and operator theory [6–10,15,1]. Group algebras were extensively studied for locally compact groups. One of the main instruments in those investigations was an existence of a Haar measure, which is characterized by such essential properties as of being left or right invariant and quasi-invariant relative left and right shifts and to the inversion on the entire group.

But substantially less is known for nonlocally compact groups. If a nontrivial Borel measure on a topological Hausdorff group quasi-invariant relative to the entire group is given, then such group is locally compact according to A. Weil's theorem. Therefore, on nonlocally compact Hausdorff groups Borel measures may be quasi-invariant relative to proper subgroups only. This is the reason of many differences between group algebras of locally compact and nonlocally compact groups. For nonlocally compact groups they are already nonassociative. This work continues previous publications of the author.

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In this article families of topological groups which may be nonlocally compact are considered. Group algebras of nonlocally compact Hausdorff topological groups are studied. Particularly, operators on nonlocally compact group algebras and their isomorphisms are investigated. Borel regular radonian measures μ_α on topological groups G_α quasi-invariant relative to dense subgroups G_β are taken. The Radon and Borel regularity properties for measures are not very restrictive (see Chapter 1 in [2] and Chapter 2 in [5]). The constructions of such measures were described in [2,3,11–14] and references therein.

The main results of this paper are obtained for the first time and are contained in Theorems 11, 15, 16, 18.

2. Group algebra

Definition 1. Let Λ be a directed set and $\{G_\alpha\colon \alpha \in \Lambda\}$ be a family of topological groups with completely regular (i.e. $T_1 \cap T_{3\frac{1}{2}}$) topologies τ_α such that

(1) $\theta_\alpha^\beta : G_\beta \to G_\alpha$ is a continuous algebraic embedding with continuous inverse $(\theta_\alpha^\beta)^{-1}$, $\theta_\alpha^\beta(G_\beta)$ is a proper subgroup in G_α for each $\alpha < \beta \in \Lambda$;
(2) $\tau_\alpha \cap \theta_\alpha^\beta(G_\beta) \subset \theta_\alpha^\beta(\tau_\beta)$ and $\theta_\alpha^\beta(G_\beta)$ is dense in G_α for each $\alpha < \beta \in \Lambda$;
(3) G_α is complete relative to the left uniformity with entourages of the diagonal of the form $\mathcal{U} = \{(h, g)\colon h, g \in G_\alpha;\ h^{-1}g \in U\}$ with neighborhoods U of the unit element e_α in G_α, $U \in \tau_\alpha$, $e_\alpha \in U$;
(4) for each $\beta = \phi(\alpha)$ the embedding θ_α^β is precompact, that is by our definition for every open set U in G_β containing the unit element e_β a neighborhood $V \in \tau_\beta$ of e_β exists so that $V \subset U$ and $\theta_\alpha^\beta(V)$ is precompact in G_α, i.e. its closure $cl(\theta_\alpha^\beta(V))$ in G_α is compact, where $\phi : \Lambda \to \Lambda$ is an increasing marked mapping.

Definition 2. Suppose that

(1) $\mu_\alpha : \mathcal{B}(G_\alpha) \to [0, 1]$ is a probability measure on the Borel σ-algebra $\mathcal{B}(G_\alpha)$ of a group G_α from Definition 1 with $\mu_\alpha(G_\alpha) = 1$ so that
(2) μ_α is quasi-invariant relative to the left and right shifts on $h \in \theta_\alpha^\beta(G_\beta)$ for each $\alpha < \beta \in \Lambda$, where $\rho^r_{\mu_\alpha}(h, g) = (\mu_\alpha^h)(dg)/\mu(dg)$ and $\rho^l_{\mu_\alpha}(h, g) = (\mu_{\alpha_h})(dg)/\mu(dg)$ denote quasi-invariance μ_α-integrable factors, $\mu_\alpha^h(S) = \mu(Sh^{-1})$ and $\mu_{\alpha,h}(S) = \mu_\alpha(h^{-1}S)$ for each Borel subset S in G_α. Moreover,
(3) let a density $\psi_\alpha(g) = \mu_\alpha(dg^{-1})/\mu_\alpha(dg)$ relative to the inversion exist and let it be μ_α-integrable.

A subset E in G_α has μ_α-measure zero, if a Borel subset F in G_α exists such that $E \subset F$ and $\mu_\alpha(F) = 0$. The completion of $\mathcal{B}(G_\alpha)$ by all μ_α-zero sets will be denoted by $\mathcal{A}(G_\alpha)$. The measure μ_α has the extension $\nu_\alpha : 2^{G_\alpha} \to [0, 1]$ such that $\nu_\alpha(E) := \inf\{\mu_\alpha(F)\colon E \subset F$ and $F \in \mathcal{B}(G_\alpha)\}$, where 2^{G_α} denotes the family of all subsets in G_α. The measure ν_α is Borel regular, that is, by the definition all open subsets in G_α are ν_α-measurable and each subset E in G_α is contained in a Borel subset F so that $\nu_\alpha(E) = \nu_\alpha(F)$. Evidently, $\nu_\alpha(F) = \mu_\alpha(F)$ for each Borel subset F in G_α, so ν_α on 2^{G_α} will also be denoted by μ_α.