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Bull. Sci. math. 132 (2008) 218–231

BULLETIN DES SCIENCES MATHÉMATIQUES

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Hopf bifurcation for degenerate singular points of multiplicity $2n-1$ in dimension 3

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Received 3 January 2007; accepted 9 January 2007

Available online 12 March 2007

Abstract

The main purpose of this paper is to study the Hopf bifurcation for a class of degenerate singular points of multiplicity $2n-1$ in dimension 3 via averaging theory. More specifically, we consider the system

$$\begin{aligned}
\dot{x} &= -H_y(x,y) + P_{2n}(x,y,z) + \varepsilon P_{2n-1}(x,y),\\
\dot{y} &= H_x(x,y) + Q_{2n}(x,y,z) + \varepsilon Q_{2n-1}(x,y),\\
\dot{z} &= R_{2n}(x,y,z) + \varepsilon c z^{2n-1},
\end{aligned}$$

where

$$\begin{aligned}
H &= \frac{1}{2n}\left(x^{2l}+y^{2l}\right)^m, \quad n = lm,\\
P_{2n-1} &= x\left(p_1 x^{2n-2} + p_2 x^{2n-3} y + \cdots + p_{2n-1} y^{2n-2}\right),\\
Q_{2n-1} &= y\left(p_1 x^{2n-2} + p_2 x^{2n-3} y + \cdots + p_{2n-1} y^{2n-2}\right),
\end{aligned}$$

and P_{2n}, Q_{2n} and R_{2n} are arbitrary analytic functions starting with terms of degree $2n$. We prove using the averaging theory of first order that, moving the parameter ε from $\varepsilon = 0$ to $\varepsilon \neq 0$ sufficiently small, from the origin it can bifurcate $2n-1$ limit cycles, and that using the averaging theory of second order from the origin it can bifurcate $3n-1$ limit cycles when $l = 1$.

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1 The author has been supported by the grants MCYT-Spain MTM2005-06098-C02-01 and CIRIT-Spain 2005SGR 00550.

2 The author has been supported by a CRM grant.

0007-4497/$ – see front matter
doi:10.1016/j.bulsci.2007.01.003

MSC: 37G15; 37D45

Keywords: Limit cycles; Hopf bifurcation; Averaging theory

1. Introduction and statement of the main results

Let

$$\dot{x} = P(x, y, z), \qquad \dot{y} = Q(x, y, z), \qquad \dot{z} = R(x, y, z),$$

be an analytic system in $\mathbb{R}^3$ starting with terms in P, Q and R of order $2n-1$. Then, here we say that the singular point at the origin of $\mathbb{R}^3$ has *multiplicity* $2n-1$.

The main purpose of this paper is to study the Hopf bifurcation which takes place at the singular point located at the origin for a subclass analytic differential equations in $\mathbb{R}^3$ of the form

$$\begin{aligned}
\dot{x} &= \overline{P}_{2n-1}(x, y, z) + \overline{P}_{2n}(x, y, z),\\
\dot{y} &= \overline{Q}_{2n-1}(x, y, z) + \overline{Q}_{2n}(x, y, z),\\
\dot{z} &= \overline{R}_{2n-1}(x, y, z) + \overline{R}_{2n}(x, y, z),
\end{aligned} \tag{1}$$

where $\overline{P}_{2n-1}$, $\overline{Q}_{2n-1}$ and $\overline{R}_{2n-1}$ are homogeneous polynomials of degree $2n-1$, and $\overline{P}_{2n}$, $\overline{Q}_{2n}$ and $\overline{R}_{2n}$ are analytical functions starting with terms of order $2n$.

In general Hopf bifurcation is well studied for singular points which have an eigenvalue of the form $\alpha(\varepsilon) \pm \beta(\varepsilon)i$ with $\alpha(0) = 0$ and $\alpha'(0) \neq 0$. Also in dimension 2 the Hopf bifurcation can be obtained for the singular points having eigenvalues of the form $\pm\beta i$ using the so-called Lyapunov constants, see for instance [1,2]. But for systems (1) with $n > 1$, the singular point located at the origin is degenerated and all its eigenvalues are zero, so the standard techniques for studying the limit cycles that bifurcate from the origin changing a parameter cannot be applied.

We shall study the Hopf bifurcation of a subclass of systems (1) using the averaging theory, see Section 2. Our main results are as follows.

Theorem 1. *We consider the differential systems in* $\mathbb{R}^3$ *given by*

$$\begin{aligned}
\dot{x} &= -H_y(x, y) + P_{2n}(x, y, z) + \varepsilon P_{2n-1}(x, y),\\
\dot{y} &= H_x(x, y) + Q_{2n}(x, y, z) + \varepsilon Q_{2n-1}(x, y),\\
\dot{z} &= R_{2n}(x, y, z) + \varepsilon c z^{2n-1},
\end{aligned} \tag{2}$$

where

$$\begin{aligned}
H &= \frac{1}{2n}\left(x^{2l} + y^{2l}\right)^m, \quad n = lm,\\
P_{2n-1} &= x\left(p_1 x^{2n-2} + p_2 x^{2n-3} y + \cdots + p_{2n-1} y^{2n-2}\right),\\
Q_{2n-1} &= y\left(p_1 x^{2n-2} + p_2 x^{2n-3} y + \cdots + p_{2n-1} y^{2n-2}\right),
\end{aligned}$$

and P_{2n}, Q_{2n} *and* R_{2n} *are arbitrary analytic functions starting with terms of degree* $2n$. *Then, moving the parameter* ε *from* $\varepsilon = 0$ *to* $\varepsilon \neq 0$, *from the origin of system* (2) *it can bifurcate* $2n-1$ *limit cycles.*