Bull. Sci. math. 130 (2006) 330–336

BULLETIN DES SCIENCES MATHÉMATIQUES

www.elsevier.com/locate/bulsci

Some properties of non-compact complete Riemannian manifolds

Li Ma

Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China

Received 3 December 2005; accepted 19 December 2005

Available online 3 February 2006

Abstract

In this paper, we study the volume growth property of a non-compact complete Riemannian manifold M. We improve the volume growth theorem of Calabi (1975) and Yau (1976), Cheeger, Gromov and Taylor (1982). Then we use our new result to study gradient Ricci solitons. We also show that on M, for any $q \in (0, \infty)$, every non-negative L^q subharmonic function is constant under a natural decay condition on the Ricci curvature.

MSC: 53Cxx

Keywords: Volume growth; Gradient estimate; Ricci soliton

1. Introduction

The motivation for this paper comes from the interest in the understanding the Ricci solitons [8]. However, at this moment, almost all works in this direction are about Gradient Ricci Solitons. See [2,4,8,11]. Generally speaking, a non-compact Ricci soliton may not be a gradient Ricci soliton. So it may be interesting to consider problems related to Ricci solitons.

In this paper, we consider the volume growth properties of the non-compact complete Riemannian manifold (M, g) under a natural Ricci curvature condition. We can improve the volume growth theorem of Calabi [3] and Yau [14]. Then we use our new result to study gradient Ricci soliton. We also show that on M, for any $q \in (0, \infty)$, every non-negative L^q subharmonic function is constant under a natural decay condition on the Ricci curvature.

E-mail address: lma@math.tsinghua.edu.cn (L. Ma).

0007-4497/$ – see front matter
doi:10.1016/j.bulsci.2005.12.002

Result about Gradient Solitons is stated in section four. In section two, we generalize the result of Calabi [3] and Yau [14] on infinite volume property for Riemannian manifolds with non-negative Ricci curvature. Calabi and Yau's result was generalized by Cheeger–Gromov–Taylor (see Theorem 4.9 in [6]) to Riemannian manifolds with lower bound like

$$\mathrm{Rc} \geqslant -\frac{\nu_n}{\rho^2(x)},$$

for $\rho(x) \gg 1$ and some restricted dimensional constant ν_n, where Rc is the Ricci curvature of the metric g, and $\rho(x)$ is the distance function from some fixed point p. We can remove this restriction to the dimensional constant ν_n. Our result is

Theorem 1. *Let (M, g) be a complete non-compact Riemannian manifold. Assume that its Ricci curvature has the lower bound*

$$\mathrm{Rc}(g) \geqslant -C\rho^{-2}(x),$$

for $\rho(x) \gg 1$, where C is a constant. Then for any $x \in M$ with $\rho(x) \gg 1$ and $r > 1$, there is a constant $C(n, \mathrm{Vol}\, B_1(x))$ such that it holds

$$\mathrm{Vol}\, B_r(x) \geqslant C\big(n, \mathrm{Vol}\, B_1(x)\big)r.$$

At the first glance, one may think that Theorem 1 is false by looking at the following example: Let $M = R^2$ with metric

$$g = \mathrm{d}r^2 + \frac{1}{r^2}\,\mathrm{d}\theta^2, \quad r \geqslant 1,$$

which is smoothly extended to $r \leqslant 1$. In this case, we have

$$K = -2r^{-2}, \qquad \mathrm{d}A = r^{-1}\,\mathrm{d}r\,\mathrm{d}\theta, \quad r \geqslant 1.$$

Hence, $V(B_r(0)) \sim \ln r$, $r \to +\infty$. Actually, this corresponds to $p = 0$ and

$$(R-1)\,\mathrm{Vol}\big(B_{R+1}(p) - B_{R-1}(p)\big) \leqslant 2(n+1+C)\,\mathrm{Vol}\, B_{R+1}(p)$$

in Section 2 below.

In section three, we give some remarks on Yau's gradient estimate and vanishing properties for subharmonic functions. We may use the letter C to denote various constants in different places.

2. Proof of Theorem 1

We let D and $R(.,.)$. be the Levi-Civita derivative and Riemannian curvature of the metric g respectively. Let $\gamma(t)$ be the minimizing geodesic from p to a point x, parametrized by the arc-length parameter such that $\gamma(0) = p$ and $\gamma(r) = x$. Assume that the point x is inside the cut locus of $p \in M$. Let Y_0 be a vector in T_xM with $g(Y_0, \frac{\partial}{\partial t}) = 0$. Then we can get an Jacobi field Y by extending Y_0 along γ (see [1] or [5]). Let $I_0^r(.,.)$ be the index form along γ. Then the Hessian of ρ at x

$$H(\rho)(Y_0, Y_0) = YY\rho - D_Y Y\rho$$

can be written as

$$\int_0^r \big(|D_tY|^2 - g\big(R(Y, D_t)D_t, Y\big)\big)\,\mathrm{d}t$$