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Some remarks about 1-convex manifolds on which all holomorphic line bundles are trivial

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Abstract

We give an abridged proof of an example already considered in [M. Colţoiu, On 1-convex manifolds with 1-dimensional exceptional set, Rev. Roumaine Math. Pures et Appl. 43 (1998) 97–104] of a 1-convex manifold X of dimension 3 such that all holomorphic line bundles on X are trivial. We also point out several mistakes of [Vo Van Tan, On the quasiprojectivity of compactifiable strongly pseudoconvex manifolds, Bull. Sci. Math. 129 (2005) 501–522] concerning this topic.

Keywords: 1-convex manifold; Moishezon manifold

In [3] it was proved, using an example due to B.G. Moishezon [8] the following theorem:

Theorem 1. *There exists a* 1*-convex manifold X of dimension* 3 *with exceptional set* $\mathbb{P}^1$ *such that all holomorphic line bundles on X are trivial.*

This shows that there is not a natural analogy with Moishezon manifolds because clearly on each Moishezon manifold there are non-trivial holomorphic line bundles.

We give an abridged proof of Theorem 1 (as in [3]).

As in [8] let $Y \subset \mathbb{P}^4$ be a hypersurface of degree $d > 2$ having only one singular point $p \in Y$ which is a quadratic non-degenerate singular point. Consider $\sigma : V \to \mathbb{P}^4$ to be the blowing-up of $\mathbb{P}^4$ at p, let $\Sigma = \mathbb{P}^3 = \sigma^{-1}(p)$ be the exceptional divisor of V, B the proper transform of Y by σ and $T = \Sigma \cap B$. Note that B is a smooth hypersurface in V and $T = \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ is a quadric.

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We denote by L the line bundle corresponding to the divisor B. Since $d > 2$ it follows as in [3] that L is positive on V. Let now A be a linear hyperplane in V which does not meet Σ but its intersection with B is transversal. Then the pair $(V \setminus A, B \setminus A)$ is 3-connected. This follows from [6] (see condition (C)) because one considers the two strata $V \setminus B$ and B (see [3] for details). This also follows from the work of Hamm and Lê [5] because for a "good neighborhood" W of B, $B \setminus A$ is a deformation retract of $W \setminus A$, therefore $(V \setminus A, B \setminus A)$ is 3-connected. In particular the restriction map $H^2(V \setminus A, \mathbb{Z}) \to H^2(B \setminus A, \mathbb{Z})$ is bijective. One gets that $H^2(B \setminus A, \mathbb{Z}) = \mathbb{Z}$ and this group is generated by the Chern class of the line bundle corresponding to the divisor $T \subset B \setminus A$. Denote $M = B \setminus A$. Then M is 1-convex with exceptional set $T = \mathbb{P}^1 \times \mathbb{P}^1$. We consider the projection $\pi_1 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 = S$ and $\tilde{\pi} : M \to X$ with $\tilde{\pi}|_T = \pi$ such that $\tilde{\pi}$ is the blowing up of X along S. Such a contraction exists according to S. Nakano [10]. One has exactly as in [3] that $H^2(X, \mathbb{Z}) = 0$ and since p is a rational singularity [7] it follows from the exponential exact sequence that $H^1(X, \mathcal{O}^*) = 0$, therefore X has only trivial holomorphic line bundles, as claimed. One gets also from the quasiprojective theorems that M and X are simply connected.

Notice that it is possible to relax the condition of transversality of A and B with the weaker condition $A \cap B$ is smooth and instead of using the quasiprojective Lefschetz theorems we can use only the classical Lefschetz theorem (in the projective case) and Thom's isomorphism. Indeed one has the two exact sequences:

$$\begin{aligned}\mathbb{Z} &= H^0(A, \mathbb{Z}) \\ &= H^2(V, V \setminus A, \mathbb{Z}) \to H^2(V, \mathbb{Z}) \to H^2(V \setminus A, \mathbb{Z}) \to H^3(V, V \setminus A, \mathbb{Z}) \\ &= H^1(A, \mathbb{Z}) = 0\end{aligned}$$

and

$$\begin{aligned}\mathbb{Z} &= H^0(A \cap B, \mathbb{Z}) \\ &= H^2(B, B \setminus A, \mathbb{Z}) \to H^2(B, \mathbb{Z}) \to H^2(B \setminus A, \mathbb{Z}) \to H^3(B, B \setminus A, \mathbb{Z}) \\ &= H^1(A \cap B, \mathbb{Z}) = 0\end{aligned}$$

and of course the natural restriction maps between these 2 sequences. By a diagram chasing, using the classical Lefschetz theorem, one gets the isomorphism $\mathbb{Z} = H^2(V \setminus A, \mathbb{Z}) \to H^2(B \setminus A, \mathbb{Z})$. As we already remarked in this gives immediately that $H^2(X, \mathbb{Z}) = 0$, therefore $H^1(X, \mathcal{O}^*) = 0$.

Remark 2. In his recent paper [14] Vo Van Tan proved several results concerning the quasi-projectivity of compactifiable strongly pseudoconvex manifolds. Among his main results is Counterexample 1.7 in which the author shows that the example considered in my paper [3] (which is essentially based on a construction due to Moishezon [8]) of a 1-convex manifold X on which all holomorphic line bundles are trivial, is not correct. More precisely Vo Van Tan shows that in the situation considered in [14] one has $H^1(X, \mathcal{O}^*) = \mathbb{Z}/p\mathbb{Z}$ if $p \in \mathbb{N}$. He shows that for every $p \in \mathbb{N}$ there is a suitable X of this type satisfying this equality. One gets in this way the contradiction that every natural number is equal to 0 (see also the contribution of G.K. Sankaran [12]).

Then arises the natural question: where is the mistake?

This can be explained as follows:

At pp. 506, 507 Vo Van Tan inserts paragraphs from the paper of G. Ceresa and A. Collino [2], which do not have any connection with the subject (and the paper [2] is not mentioned in