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Central limit theorem for constrained Poisson systems

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Abstract

We prove that for a class of constrained Poisson white noise fields, the scaling (continuum) limit exists and equals Gaussian white noise, indexed by mean zero test functions. Under natural conditions on the Lévy measure, the (Poisson) moments converge to their Gaussian counterparts.

Résumé

Nous démontrons, pour une classe de champs bruit blancs contraints, que la limite continue (d'échelle) existe et dèfinit un bruit blanc Gaussien, indexé par des fonctions test centrès. Sous des conditions naturelles pour la mesure de Lévy, les moments (Poissoniens) convergent vers leurs correspondants Gaussiens.

1. Introduction

The background to this paper is the possibility to obtain modular forms [6] as moments of random fields on tori evaluated on points of finite order, see [1]. Consider a random field equation

$$LX = \xi$$

where L is a linear partial differential operator (in $\mathbb{R}^d$) with constant coefficients and ξ is a given white noise of Poisson type, say $\xi = \sum \alpha_i \delta_{x_i}$. A formal solution is obtained by defining

$$X = \sum \alpha_i G(\cdot - x_i)$$

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where G is a fundamental solution to L. In the periodic case, when X is a random field on a torus, in real dimension two, and L is the Cauchy–Riemann operator $\bar\partial := \frac{1}{2}(\partial_x + \sqrt{-1}\partial_y)$:

$$\bar\partial X = \sum \alpha_i \delta_{x_i}.$$

The periodicity of X requires the constraint, or gauge condition, $\sum \alpha_i = 0$ to hold. This means that the α_i are no more independent. In particular the one-particle space has to be removed.

In a more general setting we study the convergence of constrained massless Poisson (noise) fields to Gaussian white noise indexed by test functions with vanishing mean-value ($\int f = 0$). It is also shown that the moments converge, under appropriate conditions on the associated Lévy measure.

Convergence of the noise implies convergence of $\sum \alpha_i G(\cdot - x_i)$ to the free Gaussian massless field on the torus. The same goes for convergence of the moments.

2. Convolution semi-groups

For the background in probability and Fourier analysis the reader is referred to [2–5] and [7].

Let V be a finite dimensional real vector space of dimension d. The canonical pairing $\mathrm{Hom}(V,\mathbb{R}) \times V \to \mathbb{R}$ is denoted by $(\xi, v) \mapsto \langle \xi, v\rangle$. We fix a Euclidean inner product for V. The corresponding norm is denoted by $\|\cdot\|$ and the dual norm for $\mathrm{Hom}(V,\mathbb{R})$ is also denoted by $\|\cdot\|$. As for the normalization of Lebesgue measures we shall refer to the norm $\|\cdot\|$.

2.1. Definition. For a Borel probability measure μ on V its characteristic function (Fourier transform) φ is defined by

$$\varphi(\xi) := \int_V e^{\sqrt{-1}\langle \xi, v\rangle} \mu(dv), \quad \text{for } \xi \in \mathrm{Hom}(V,\mathbb{R}).$$

Its n-fold convolution μ^{*n} is defined by

$$\mu^{*n}(A) := \mu^{\times n}\big(\{(v_1, v_2, \dots, v_n) \in V^n;\ v_1 + v_2 + \cdots + v_n \in A\}\big) \quad \text{for } A \in \mathrm{Borel}(V),$$

where $\mu^{\times n}$ stands for the n-fold product of μ with itself.

Recall that $\varphi(\cdot)^n$ is the characteristic function of μ^{*n}.

We have the following result on the behaviour of φ near the origin.

2.2. Lemma. *Suppose that μ is a Borel probability measure μ on V with finite second order moment and $\int v\mu(dv) = 0$. Then for any ε, $0 < \varepsilon < 1$, there exists $\delta > 0$ such that $|\varphi(\xi)| \leqslant 1 - \frac{1}{2}(1-\varepsilon)\int \langle \xi, v\rangle^2 \mu(dv)$ for all $\xi \in \mathrm{Hom}(V,\mathbb{R})$ with $\int \langle \xi, v\rangle^2 \mu(dv) \leqslant \delta$.*

Proof. Using $e^x = 1 + x + (\frac{1}{2} + \int_0^1 (1-t)(e^{tx} - 1)\,dt)x^2$, we see that

$$\begin{aligned}\varphi(\xi) &= 1 - \int_V \frac{1}{2}\langle \xi, v\rangle^2 \mu(dv) + \int_V \left\{ \int_0^1 (1-t)\big(1 - e^{\sqrt{-1}t\langle \xi, v\rangle}\big)\,dt\,\langle \xi, v\rangle^2 \right\} \mu(dv) \\ &= 1 - \frac{1}{2}\|\xi\|^2 + \left\langle \xi \otimes \xi, \int_{[0,1]\times V} (1-t)\big(1 - e^{\sqrt{-1}t\langle \xi, v\rangle}\big) v \otimes v\,dt\,\mu(dv) \right\rangle.\end{aligned}$$