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Complex analysis

Some properties related to a certain class of starlike functions

Quelques propriétés liées à une classe de fonctions étoilées

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A B S T R A C T

This paper considers a class Δ^* of normalized starlike functions f analytic in the open unit disk $|z| < 1$ satisfying the inequality that

$$\left|\left\{\frac{zf'(z)}{f(z)}\right\}^2 - 1\right| < 2\left|\frac{zf'(z)}{f(z)}\right|$$

in $|z| < 1$. We first show that the class $\mathcal{S}^*(q)$ (defined below) is a subclass of Δ^* and then obtain some useful properties of these classes of functions.

R É S U M É

Nous considérons dans cette Note une classe Δ^* de fonctions étoilées normalisées f, analytiques dans le disque unité ouvert $|z| < 1$ et y satisfaisant l'inégalité

$$\left|\left\{\frac{zf'(z)}{f(z)}\right\}^2 - 1\right| < 2\left|\frac{zf'(z)}{f(z)}\right|.$$

Nous montrons d'abord que la classe $S^*(q)$ (définie ci-dessous) est une sous-classe de Δ^*, puis nous obtenons quelques propriétés utiles de ces classes de fonctions.

1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the open unit disc $\mathbb{U} = \{z : |z| < 1\}$ on the complex plane $\mathbb{C}$. Also, let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ comprised of functions f normalized by $f(0) = 0$, $f'(0) = 1$, and let $\mathcal{S} \subset \mathcal{A}$ denote the class of functions that are univalent in $\mathbb{U}$. Let a function f be analytic univalent in the unit disc $\mathbb{U} = \{z : |z| < 1\}$ on the complex plane $\mathbb{C}$ with the normalization $f(0) = 0$, then f maps $\mathbb{U}$ onto a starlike domain with respect to $w_0 = 0$ if and only if

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$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad (z \in \mathbb{U}). \tag{1.1}$$

It is well known that if an analytic function f satisfies (1.1) and $f(0) = 0$, $f'(0) \neq 0$, then f is univalent and starlike in $\mathbb{U}$. The set of all functions $f \in \mathcal{A}$ that are starlike univalent in $\mathbb{U}$ will be denoted by $\mathcal{S}^*$. We say that an analytic function f is subordinate to an analytic function g, and write $f(z) \prec g(z)$, if and only if there exists a function ω, analytic in $\mathbb{U}$ such that $\omega(0) = 0$, $|\omega(z)| < 1$ for $|z| < 1$ and $f(z) = g(\omega(z))$. In particular, if g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}). \tag{1.2}$$

We will need the following basic lemma in the theory of differential subordinations.

Lemma 1.1. *(See [9], see also [10, p. 24].) Assume that $\mathcal{Q}$ is the set of analytic functions that are injective on $\overline{\mathbb{U}} \setminus E(f)$, where $E(f) := \{\zeta : \zeta \in \partial\mathbb{U}$ and $\lim_{z\to\zeta} f(z) = \infty\}$, and are such that $f'(\zeta) \neq 0$ $(\zeta \in \partial(\mathbb{U}) \setminus E(f))$. Let $\psi \in \mathcal{Q}$ with $q(0) = a$ and let $\varphi(z) = a + a_m z^m + \cdots$ be analytic in $\mathbb{U}$ with $\varphi(z) \not\equiv a$ and $m \in \mathbb{N}$. If $\varphi \not\prec \psi$ in $\mathbb{U}$, then there exist points $z_0 = r_0 e^{i\theta} \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U} \setminus E(\psi)$, for which $\varphi(|z| < r_0) \subset \psi(\mathbb{U})$, $\varphi(z_0) = \psi(\zeta_0)$ and $z_0\varphi'(z_0) = s\zeta_0\psi'(\zeta_0)$, for some $s \geq m$.*

2. Main result

Let Δ^* be defined by

$$\Delta^* = \left\{ f \in \mathcal{S}^* : \left| \left\{\frac{zf'(z)}{f(z)}\right\}^2 - 1 \right| < 2\left|\frac{zf'(z)}{f(z)}\right|, z \in \mathbb{U} \right\}. \tag{2.1}$$

We prove the following theorem, which would be used to obtain an equivalent class of Δ^*.

Theorem 2.1. *If $p(z) \in \mathcal{H}$ with $p(0) = 1$, then*

$$p(z) \prec q(z) := z + \sqrt{1 + z^2}, \quad q(0) = 1, \tag{2.2}$$

implies that $\mathfrak{Re}\{p(z)\} > 0$ and

$$\left| p^2(z) - 1 \right| < 2|p(z)|, \quad z \in \mathbb{U}. \tag{2.3}$$

Proof. We first show that $q(z)$ is univalent in $\mathbb{U}$. Assume that $q(z_1) = q(z_2)$, for some $z_1, z_2 \in \mathbb{U}$, then

$$z_1 - z_2 = \sqrt{1 + z_2^2} - \sqrt{1 + z_1^2}. \tag{2.4}$$

Upon squaring (2.4), we get

$$1 + z_1 z_2 = \sqrt{1 + z_2^2}\sqrt{1 + z_1^2}. \tag{2.5}$$

Again squaring (2.5), we are easily lead to $z_1 = z_2$, and hence, $q(z)$ is univalent in $\mathbb{U}$. Evidently, then by virtue of (1.2), the subordination (2.2) is equivalent to

$$p(\mathbb{U}) \subset q(\mathbb{U}). \tag{2.6}$$

In order to prove that $\mathfrak{Re}\{p(z)\} > 0$, $z \in \mathbb{U}$, it suffices to show that $\mathfrak{Re}\left\{q(e^{it})\right\} \geq 0$, $t \in [0, 2\pi)$. Let $z = e^{it}$, $t \in [0, 2\pi)$, then

$$e^{it} + \sqrt{e^{2it} + 1} = \begin{cases} \cos t + i\sin t + \sqrt{2\cos t}(\cos t/2 + i\sin t/2) & \text{for } t \in [0, \pi/2), \\ i & \text{for } t = \pi/2, \\ \cos t + i\sin t + \sqrt{|2\cos t|}(\sin t/2 - i\cos t/2) & \text{for } t \in (\pi/2, 3\pi/2), \\ -i & \text{for } t = 3\pi/2, \\ \cos t + i\sin t + \sqrt{2\cos t}(-\cos t/2 - i\sin t/2) & \text{for } t \in (3\pi/2, 2\pi). \end{cases} \tag{2.7}$$

Now, some simple calculations show that $\mathfrak{Re}\left\{e^{it} + \sqrt{e^{2it} + 1}\right\} = 0$, if and only if $t = \pi/2$, or if $t = 3\pi/2$, which implies that $\mathfrak{Re}\{q(z)\} > 0$ in $\mathbb{U}$ (see Fig. 1). From (2.7), we can also find that $q(e^{it})$ is a union of two circular arcs: smaller arc $|w + 1| = \sqrt{2}$ and greater arc $|w - 1| = \sqrt{2}$.

Finally, we will prove (2.3). It follows from (2.2) that

$$(p(z) - w(z))^2 = 1 + w^2(z) \tag{2.8}$$