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Extension formulas and deformation invariance of Hodge numbers

Formules d'extension et invariance par déformation des nombres de Hodge

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ABSTRACT

We introduce a canonical isomorphism from the space of pure-type complex differential forms on a compact complex manifold to the one on its infinitesimal deformations. By use of this map, we generalize an extension formula in a recent work of K. Liu, X. Yang and the second author. As a direct corollary of the extension formulas, we prove several deformation invariance theorems for Hodge numbers on some certain classes of complex manifolds, without using the Frölicher inequality or the topological invariance of the Betti numbers.

RÉSUMÉ

Nous introduisons un isomorphisme canonique entre l'espace des formes différentielles complexes de type pur sur une variété complexe, compacte, et celui de ses déformations infinitésimales, et nous l'utilisons pour généraliser la formule d'extension récemment obtenue par K. Liu, X. Yang et le second auteur. Comme corollaire direct des formules d'extension, nous établissons plusieurs théorèmes d'invariance par déformation des nombres de Hodge des variétés complexes, sans avoir recours à l'inégalité de Frölicher ou à l'invariance topologique des nombres de Betti.

1. Introduction and main results

This paper aims at studying the deformation invariance of Hodge numbers using an iteration method to construct an explicit extension of Dolbeault cohomology classes.

Let $\pi : \mathfrak{X} \to \triangle$ be a holomorphic family of n-dimensional compact complex manifolds with the central fiber $\pi^{-1}(0) = X_0$ and its infinitesimal deformations $\pi^{-1}(t) = X_t$, where $\triangle$ is a small disk in $\mathbb{C}$ for simplicity. Then there exists a transversely

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holomorphic trivialization $F_\sigma : \mathfrak{X} \xrightarrow{(\sigma,\pi)} X_0 \times \triangle$ (cf. [20, Proposition 9.5] and [3, Appendix A]), which gives us the Kuranishi data $\varphi(t)$ (or φ), depending holomorphically on t, with the integrability

$$\overline{\partial}\varphi(t) = \frac{1}{2}[\varphi(t), \varphi(t)]. \tag{1.1}$$

Fix an open coordinate covering $\left\{\mathfrak{U} : (w_j^\alpha, t) \in U^\alpha, U^\alpha \in \mathfrak{U}\right\}$ of $\mathfrak{X}$, with a restricted covering $\left\{\mathfrak{U}_0 : z_j^\alpha \in U_0^\alpha := U^\alpha \bigcap X_0, U^\alpha \bigcap X_0 \in \mathfrak{U}_0\right\}$ of X_0. As we focus on one coordinate chart, the superscript α is suppressed. As in [3,10,9], the operator e^{i_φ} is defined by

$$e^{i_\varphi} = \sum_{k=0}^{\infty} \frac{1}{k!} i_\varphi^k,$$

where i_φ^k denotes k times of the contraction operator $i_\varphi = \varphi\lrcorner$ and $e^{i_{\overline{\varphi}}}$ is similarly defined. It is known that $\{e^{i_\varphi}\left(dz^i\right)\}_{i=1}^n$ and $\{\overline{e^{i_\varphi}\left(dz^i\right)}\}_{i=1}^n$ are the local bases of $T_{X_t}^{*(1,0)}$ and $T_{X_t}^{*(0,1)}$, respectively. Inspired by these, we introduce:

Definition 1.1. A canonical map between $A^{p,q}(X_0)$ and $A^{p,q}(X_t)$ is defined as:

$$\begin{array}{rccc} e^{i_\varphi | i_{\overline{\varphi}}}: & A^{p,q}(X_0) & \to & A^{p,q}(X_t) \\ & \omega & \mapsto & e^{i_\varphi | i_{\overline{\varphi}}}(\omega), \end{array}$$

where

$$e^{i_\varphi | i_{\overline{\varphi}}}(\omega) = \sum_{\substack{i_1,\cdots,i_p \\ j_1,\cdots,j_q}} \frac{1}{p!q!} \omega_{i_1,\cdots,i_p;j_1,\cdots,j_q}(z) \left(e^{i_\varphi}\left(dz^{i_1} \wedge \cdots \wedge dz^{i_p}\right)\right) \wedge \left(e^{i_{\overline{\varphi}}}\left(d\overline{z}^{j_1} \cdots \wedge d\overline{z}^{j_q}\right)\right)$$

and ω is locally written as $\sum_{\substack{i_1,\cdots,i_p \\ j_1,\cdots,j_q}} \frac{1}{p!q!} \omega_{i_1,\cdots,i_p;j_1,\cdots,j_q}(z) dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\overline{z}^{j_1} \wedge \cdots d\overline{z}^{j_q}$.

It is easy to check that $e^{i_\varphi | i_{\overline{\varphi}}}$ is independent of the choice of the local coordinates and is actually a real isomorphism. From the explicit formula of φ (cf. [11, pp. 150]), a careful calculation yields:

Lemma 1.2.

$$\begin{cases} dw^\alpha & = \frac{\partial w^\alpha}{\partial z^i}\left(e^{i_\varphi}\left(dz^i\right)\right) \\ \frac{\partial}{\partial w^\alpha} & = \left((\mathbb{1} - \varphi\overline{\varphi})^{-1}\left(\frac{\partial w}{\partial z}\right)^{-1}\right)_\alpha^j \frac{\partial}{\partial z^j} - \left((\mathbb{1} - \overline{\varphi}\varphi)^{-1}\overline{\varphi}\left(\frac{\partial w}{\partial z}\right)^{-1}\right)_\alpha^j \frac{\partial}{\partial \overline{z}^j}, \end{cases}$$

where $\overline{\varphi}\varphi := \varphi\lrcorner\overline{\varphi}$ and $\varphi\overline{\varphi}$ is similarly defined.

Corollary 1.3. $\frac{\partial w^\alpha}{\partial z^i}\frac{\partial}{\partial w^\alpha} = \left((\mathbb{1} - \varphi\overline{\varphi})^{-1}\right)_i^j \frac{\partial}{\partial z^j} - \left((\mathbb{1} - \overline{\varphi}\varphi)^{-1}\overline{\varphi}\right)_i^j \frac{\partial}{\partial \overline{z}^j}$.

Then we get the following useful local formula:

Lemma 1.4.

$$\begin{aligned} d\left(e^{i_\varphi}\left(dz^i\right)\right) = & \left((\mathbb{1} - \overline{\varphi}\varphi)^{-1}\overline{\varphi}\right)_{\bar{k}}^{\bar{l}} \frac{\partial \varphi_{\bar{l}}^i}{\partial z^j}\left(\overline{e^{i_\varphi}\lrcorner dz^k}\right) \wedge \left(e^{i_\varphi}\lrcorner dz^j\right) \\ & - \left((\mathbb{1} - \overline{\varphi}\varphi)^{-1}\right)_{\bar{k}}^{\bar{l}} \frac{\partial \varphi_{\bar{l}}^i}{\partial z^j}\left(\overline{e^{i_\varphi}\lrcorner dz^k}\right) \wedge \left(e^{i_\varphi}\lrcorner dz^j\right), \end{aligned}$$

which describes the d-operator under the local frames $\{e^{i_\varphi}\left(dz^i\right), \overline{e^{i_\varphi}\left(dz^i\right)}\}_{i=1}^n$.

Using these, one has:

Proposition 1.5. *Let f be a smooth function on X_0. Then*

$$df = e^{i_\varphi | i_{\overline{\varphi}}}\Big((\mathbb{1} - \varphi\overline{\varphi})^{-1}\lrcorner(\partial - \overline{\varphi}\lrcorner\overline{\partial})f + (\mathbb{1} - \overline{\varphi}\varphi)^{-1}\lrcorner(\overline{\partial} - \varphi\lrcorner\partial)f\Big).$$