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Digital homotopy fixed point theory

Théorie du point fixe pour les homotopies digitales

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ABSTRACT

In this paper, we construct a framework which is called the digital homotopy fixed point theory. We get new results associating digital homotopy and fixed point theory. We also give an application on this theory.

RÉSUMÉ

Nous démontrons de nouveaux résultats sur les images digitales dont les homotopies digitales entre deux transformations continues de l'image possèdent un chemin de points fixes. Ceci conduit à une théorie du point fixe des homotopies digitales, dont nous donnons une application sur une image digitale.

1. Introduction

Digital topology plays a key role in image processing and computer graphics. In this field, the general aim is to obtain significant results on digital images in $\mathbb{Z}^n$ by using methods of geometric and algebraic topology. Fixed point theory interacts with several areas of mathematics such as mathematical analysis, general topology, and functional analysis. There are various applications of fixed point theory in mathematics, topology, game theory, computer science, engineering, and image processing. Fixed point theorems are used to solve some problems in mathematics and engineering.

In recent years, there have been many developments in digital topology. Boxer [1–6] studied digital images. Some results and characteristic properties on the digital homology groups of $2D$ digital images are given in [7] and [14]. Ege and Karaca [8] construct Lefschetz fixed point theory for digital images and study the fixed point properties of digital images. Ege and Karaca [9] give relative and reduced Lefschetz fixed point theorems for digital images. They also calculate degree of the antipodal map for sphere-like digital images using fixed point properties. Ege and Karaca [10] prove Banach's fixed point theorem for digital images and give an application to image processing.

This work is organized as follows. In the first part, we give the required background about the digital topology and digital homotopy. Then, we state and prove some results on digital retractions and digital fixed point theory. We give an application of digital fixed point theory to a digital image.

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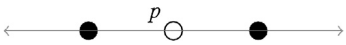

Fig. 1. 2-Adjacency.

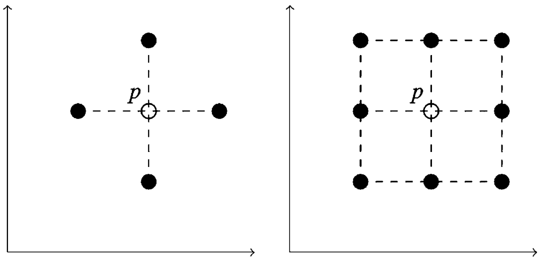

Fig. 2. 4-Adjacency and 8-adjacency.

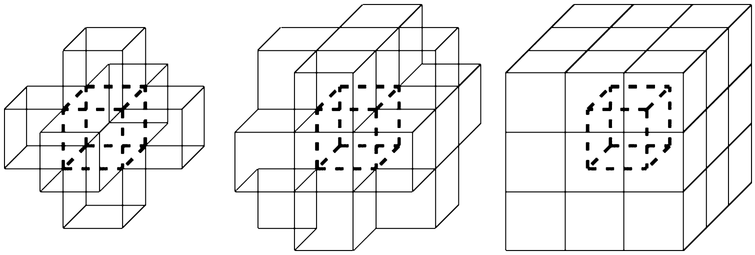

Fig. 3. 6-Adjacency, 18-adjacency, and 26-adjacency.

2. Preliminaries

A digital image is a pair (X, κ), where $X \subseteq \mathbb{Z}^n$ for some positive integer n and κ is an adjacency relation for the members of X. There are various adjacency relations [12,13].

Definition 2.1. (See [4].) For a positive integer l with $1 \le l \le n$ and two distinct points $p = (p_1, p_2, \ldots, p_n)$, $q = (q_1, q_2, \ldots, q_n) \in \mathbb{Z}^n$, p and q are c_l*-adjacent* if

(1) there are at most l indices i such that $|p_i - q_i| = 1$, and
(2) for all other indices j such that $|p_j - q_j| \neq 1$, $p_j = q_j$.

The notation c_l represents the number of points $q \in \mathbb{Z}^n$ that are adjacent to a given point $p \in \mathbb{Z}^n$. Thus, in $\mathbb{Z}$, we have $c_1 = 2$-adjacency (see Fig. 1); in $\mathbb{Z}^2$, we have $c_1 = 4$-adjacency and $c_2 = 8$-adjacency (see Fig. 2); in $\mathbb{Z}^3$, we have $c_1 = 6$-adjacency, $c_2 = 18$-adjacency, and $c_3 = 26$-adjacency [4] (see Fig. 3).

Given a natural number l in conditions (1) and (2) with $1 \le l \le n$, l determines each one of the κ-adjacency relations of $\mathbb{Z}^n$ in terms of (1) and (2) [11] as follows:

$$\kappa \in \left\{2n\ (n \ge 1),\ 3^n - 1\ (n \ge 2),\ 3^n - \sum_{t=0}^{r-2} C_t^n 2^{n-t} - 1\ (2 \le r \le n-1, n \ge 3)\right\}$$

where $C_t^n = \frac{n!}{(n-t)!t!}$.

Definition 2.2. (See [1].) The set $[a, b]_{\mathbb{Z}} = \{z \in \mathbb{Z} \mid a \le z \le b\}$ is called *a digital interval* where $a, b \in \mathbb{Z}$ and $a < b$.

Definition 2.3. (See [12].) Given two points $x_i, y_i \in (X_i, \kappa_i)$, $i \in \{0, 1\}$, (x_0, x_1) and (y_0, y_1) are adjacent in $X_0 \times X_1$ if and only if one of the following is satisfied:

(i) $x_0 = y_0$ and x_1 and y_1 are κ_1-adjacent; or
(ii) x_0 and y_0 are κ_0-adjacent and $x_1 = y_1$; or
(iii) x_0 and y_0 are κ_0-adjacent and x_1 and y_1 are κ_1-adjacent.

The adjacency of the Cartesian product of digital images (X_0, κ_0) and (X_1, κ_1) is denoted by κ_*.