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Numerical analysis

Solving the mixed Sylvester matrix equations by matrix decompositions

Résolution d'équations matricielles de Sylvester mixtes par décompositions de matrices

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ABSTRACT

By applying the generalized singular-value decompositions (GSVDs) of matrix pairs, a necessary and sufficient solvability condition for mixed Sylvester equations is established, the explicit representation of the general solution is given. Also, the minimum-norm solution of the matrix equations is discussed.

RÉSUMÉ

En utilisant les décompositions en valeurs singulières généralisées (GSVDs) de couples de matrices, on établit une condition nécessaire et suffisante de résolubilité d'équations de Sylvester mixtes et on donne une représentation explicite de la solution générale. On étudie également la solution de norme minimale d'équations matricielles.

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1. Introduction

The purpose of this work is to study the so-called mixed Sylvester matrix equations

$$A_1X - YB_1 = C_1,\ A_2Z - YB_2 = C_2, \tag{1}$$

where $A_1 \in \mathbf{C}^{m\times n}$, $B_1 \in \mathbf{C}^{l\times q}$, $C_1 \in \mathbf{C}^{m\times q}$, $A_2 \in \mathbf{C}^{m\times p}$, $B_2 \in \mathbf{C}^{l\times d}$ and $C_2 \in \mathbf{C}^{m\times d}$.

Eqs. (1) can also be equivalently written as (in matlab notation):

$$\mathrm{blkdiag}(A_1, A_2)\mathrm{blkdiag}(X, Z) - \mathrm{kron}(I_2, Y)\mathrm{blkdiag}(B_1, B_2) = \mathrm{blkdiag}(C_1, C_2).$$

There are some valuable works on formulating solutions to the mixed Sylvester matrix Eqs. (1). For example, Liu [5] derived a solvability condition of (1) by using the ranks of matrices. By applying the equivalence of matrices, Lee and Vu [4] showed that Eqs. (1) are consistent if and only if there exist invertible matrices R_1, R_2 and S such that

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$$\begin{bmatrix} A_1 & C_1 \\ 0 & B_1 \end{bmatrix} R_1 = S \begin{bmatrix} A_1 & 0 \\ 0 & B_1 \end{bmatrix}, \quad \begin{bmatrix} A_2 & C_2 \\ 0 & B_2 \end{bmatrix} R_2 = S \begin{bmatrix} A_2 & 0 \\ 0 & B_2 \end{bmatrix}.$$

Recently, Wang and He [8] provided some necessary and sufficient solvability conditions and the expression of the general solution of (1) by virtue of the ranks and generalized inverses of matrices.

The strategy adopted here is to use the generalized singular value decompositions (GSVDs) of matrix pairs to decouple the equations of (1) to obtain some profound results. Beginning in Section 2, we first consider some special cases where some constraints are imposed on coefficient matrices, then, for the general situation, we formulate the necessary and sufficient conditions for the existence of the solution of (1) directly by means of the generalized singular value decompositions of the matrix pairs (A_1, A_2) and (B_1, B_2), and construct the explicit representation of the general solution when it is solvable. Furthermore, we will provide the minimum-norm solution of (1) by using the expression of the general solution.

Throughout this paper, we denote the complex $m \times n$ matrix space by $\mathbf{C}^{m\times n}$, the set of all unitary matrices in $\mathbf{C}^{n\times n}$ by $\mathbf{UC}^{n\times n}$. A^{H} and A^+ stand for the conjugate transpose and the Moore–Penrose generalized inverse of a complex matrix A, respectively. I_n represents the identity matrix of size n. We define an inner product: $\langle A, B\rangle = \operatorname{trace}(B^{\mathrm{H}}A)$ for all $A, B \in \mathbf{C}^{m\times n}$, then $\mathbf{C}^{m\times n}$ is a Hilbert inner product space and the norm of a matrix generated by this inner product is Frobenius norm. For $A = [\alpha_{ij}]_{m\times n}$ and $B = [\beta_{ij}]_{m\times n}$, $A * B$ represents the Hadamard product of the matrices A and B, that is, $A * B = [\alpha_{ij}\beta_{ij}]_{m\times n}$.

2. The solution to the mixed Sylvester matrix Eqs. (1)

The following lemmata are needed in what follows.

Lemma 1. *(See [2].) If $A \in \mathbf{C}^{m\times k}$, $B \in \mathbf{C}^{l\times n}$ and $C \in \mathbf{C}^{m\times n}$, then the equation $AXB = C$ has a solution $X \in \mathbf{C}^{k\times l}$ if and only if $AA^+CB^+B = C$. In this case, the general solution of the matrix equation $AXB = C$ can be described as $X = A^+CB^+ + (I_k - A^+A)W + T(I_l - BB^+)$, where $W, T \in \mathbf{C}^{k\times l}$ are arbitrary matrices.*

Lemma 2. *(See [1].) Let $A \in \mathbf{C}^{m\times k}$, $B \in \mathbf{C}^{l\times n}$ and $C \in \mathbf{C}^{m\times n}$. The equation*

$$AX - YB = C \tag{2}$$

has a solution $X \in \mathbf{C}^{k\times n}$, $Y \in \mathbf{C}^{m\times l}$ if and only if $(I_m - AA^+)C(I_n - B^+B) = 0$. If this is the case, the general solution of (2) has the form

$$X = A^+C + A^+TB + (I_k - A^+A)W, \quad Y = -(I_m - AA^+)CB^+ + T - (I_m - AA^+)TBB^+,$$

where $W \in \mathbf{C}^{k\times n}$ and $T \in \mathbf{C}^{m\times l}$ are arbitrary matrices.

We consider some special cases.

Case 1. If A_1 is square and nonsingular, then from the first equation of (1), we can get

$$X = A_1^{-1}C_1 + A_1^{-1}YB_1. \tag{3}$$

By Lemma 2, the second equation of (1) has a solution $Z \in \mathbf{C}^{p\times d}$, $Y \in \mathbf{C}^{m\times l}$ if and only if

$$(I_m - A_2A_2^+)C_2(I_d - B_2^+B_2) = 0.$$

If this is the case, the general solution of the matrix equation $A_2Z - YB_2 = C_2$ has the form

$$Z = A_2^+C_2 + A_2^+TB_2 + (I_p - A_2^+A_2)W, \tag{4}$$

$$Y = -(I_m - A_2A_2^+)C_2B_2^+ + T - (I_m - A_2A_2^+)TB_2B_2^+, \tag{5}$$

where $W \in \mathbf{C}^{p\times d}$ and $T \in \mathbf{C}^{m\times l}$ are arbitrary matrices. Substituting (5) into (3), we have obtained the following result.

Theorem 1. *Suppose that $A_1 \in \mathbf{C}^{m\times m}$, $B_1 \in \mathbf{C}^{l\times q}$, $C_1 \in \mathbf{C}^{m\times q}$, $A_2 \in \mathbf{C}^{m\times p}$, $B_2 \in \mathbf{C}^{l\times d}$ and $C_2 \in \mathbf{C}^{m\times d}$. If A_1 is nonsingular, then the equation of (1) has a solution $X \in \mathbf{C}^{m\times q}$, $Y \in \mathbf{C}^{m\times l}$, $Z \in \mathbf{C}^{p\times d}$ if and only if $(I_m - A_2A_2^+)C_2(I_d - B_2^+B_2) = 0$. In this case, the general solution of (1) can be expressed as*

$$X = A_1^{-1}C_1 - A_1^{-1}(I_m - A_2A_2^+)C_2B_2^+B_1 + A_1^{-1}TB_1 - A_1^{-1}(I_m - A_2A_2^+)TB_2B_2^+B_1,$$

$$Y = -(I_m - A_2A_2^+)C_2B_2^+ + T - (I_m - A_2A_2^+)TB_2B_2^+, \quad Z = A_2^+C_2 + A_2^+TB_2 + (I_p - A_2^+A_2)W,$$

where $W \in \mathbf{C}^{p\times d}$ and $T \in \mathbf{C}^{m\times l}$ are arbitrary matrices.