C. R. Acad. Sci. Paris, Ser. I 353 (2015) 795–800

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Ordinary differential equations/Dynamical systems

Formal normal form of A_k slow–fast systems

Forme normale formelle des systèmes lents–rapides de type A_k

Hildeberto Jardón-Kojakhmetov

Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, P.O. Box 407, 9700 AK, Groningen, The Netherlands

A R T I C L E   I N F O

Article history:
Received 18 March 2015
Accepted after revision 25 June 2015
Available online 16 July 2015

Presented by the Editorial Board

A B S T R A C T

An A_k slow–fast system is a particular type of singularly perturbed ODE. The corresponding slow manifold is defined by the critical points of a universal unfolding of an A_k singularity. In this note we propose a formal normal form of A_k slow–fast systems.

R É S U M É

Un système lent–rapide de type A_k est une équation différentielle ordinaire singulièrement perturbée avec une structure particulière. La varieté lente correspondante est définie par les points critiques d'un déploiement universel d'une singularité de type A_k. Dans cette note, nous proposons une forme normale formelle des systèmes lents–rapides de type A_k.

1. Introduction

A slow–fast system (SFS) is a singularly perturbed ODE usually written as

$$\begin{aligned} \dot{x} &= f(x,z,\varepsilon) \\ \varepsilon\dot{z} &= g(x,z,\varepsilon) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^m$, $z \in \mathbb{R}^n$ and $0 < \varepsilon \ll 1$ is a small parameter, and where the over-dot denotes the derivative with respect to a time parameter t. Slow–fast systems are often used as mathematical models of phenomena that occur in two time scales. A couple of classical examples of real life phenomena that were modeled by an SFS are Zeeman's heartbeat and nerve-impulse models [17]. For $\varepsilon \neq 0$, we can define a new time parameter τ by $t = \varepsilon\tau$. With this new time τ, we can write (1) as

$$\begin{aligned} x' &= \varepsilon f(x,z,\varepsilon) \\ z' &= g(x,z,\varepsilon), \end{aligned} \tag{2}$$

where the prime denotes the derivative with respect to τ. An important geometric object in the study of SFSs is the *slow manifold*, which is defined by

E-mail address: h.jardon.kojakhmetov@gmail.com.

http://dx.doi.org/10.1016/j.crma.2015.06.009
1631-073X/

$$S = \left\{(x,z) \in \mathbb{R}^m \times \mathbb{R}^n \,|\, g(x,z,0) = 0\right\}. \tag{3}$$

When $\varepsilon = 0$, the manifold S serves as the phase space of (1) and as the set of equilibrium points of (2). In the rest of the document, we prefer to work with an SFS written as (2). Furthermore, to avoid working with an ε-parameter family of vector fields as in (2), we extend (2) by adding the trivial equation $\varepsilon' = 0$. To be more precise, we treat a C^∞-smooth vector field defined as follows.

Definition 1.1 *(A_k slow–fast system).* Let $k \in \mathbb{N}$ with $k \geq 2$. An A_k slow–fast system (for short A_k-SFS) is a vector field X of the form

$$X = \varepsilon(1+\varepsilon f_1)\frac{\partial}{\partial x_1} + \sum_{i=2}^{k-1} \varepsilon^2 f_i \frac{\partial}{\partial x_i} - (G_k - \varepsilon f_k)\frac{\partial}{\partial z} + 0\frac{\partial}{\partial \varepsilon}, \tag{4}$$

where $G_k = z^k + \sum_{i=1}^{k-1} x_i z^{i-1}$ and where each $f_i = f_i(x_1, \ldots, x_{k-1}, z, \varepsilon)$ is a C^∞-smooth function vanishing at the origin.

Multi-scale models described by an A_k-SFS are of interest as they exhibit local, fast transitions between stable states, e.g., [8,13,14,17].

Remark 1.1. The slow manifold associated with an A_k-SFS is defined by

$$S = \left\{(x,z) \in \mathbb{R}^k \,|\, z^k + \sum_{i=1}^{k-1} x_i z^{i-1} = 0\right\}. \tag{5}$$

The manifold S can be regarded as the critical set of the universal unfolding of a smooth function with an A_k singularity at the origin [1,3]. Hence the name A_k-SFS.

Observe that the origin is a non-hyperbolic equilibrium point of X and thus, it is not possible to study its local dynamics with the classical Geometric Singular Perturbation Theory [5]. In this case, the blow-up technique [4,9] can be applied to desingularize the SFS. This methodology has been successfully used in many cases, e.g., [2,7,10,11,15,16], where many of these deal with an A_k-SFS with fixed $k = 2$ or $k = 3$. Briefly speaking, the blow-up technique consists in an appropriate change of coordinates under which the induced vector field is regular or has simpler singularities (hyperbolic or partially hyperbolic).

In this paper we propose a normal form of A_k-SFSs given by Definition 1.1. In this normalization, the unknown functions f_i of (4) are eliminated. As it is shown below, the structure of the A_k-SFS plays an important role in the normalization process. Moreover, this normalization greatly simplifies the local analysis of systems given by (4), as shown in [6,7].

2. Formal normal form of an A_k slow–fast system

We regard the vector field X of Definition 1.1 as $X = F + P$, where F and P are smooth vector fields called "the principal part" and "the perturbation" respectively. That is:

$$F = \varepsilon\frac{\partial}{\partial x_1} + \sum_{i=2}^{k-1} 0\frac{\partial}{\partial x_i} - G_k\frac{\partial}{\partial z} + 0\frac{\partial}{\partial \varepsilon}, \qquad P = \sum_{i=1}^{k-1} \varepsilon^2 f_i \frac{\partial}{\partial x_i} + \varepsilon f_k \frac{\partial}{\partial z} + 0\frac{\partial}{\partial \varepsilon}. \tag{6}$$

The idea of the rest of the document is motivated by [12]. In short, we want to formally simplify the expression of X by eliminating the perturbation P. The terminology used below is that of [12].

The vector field F is quasihomogeneous of type $r = (k, k-1, \ldots, 1, 2k-1)$ and quasidegree $k-1$ [1,12]. From now on, we fix the type of quasihomogeneity r. A quasihomogeneous object of type r will be called r-quasihomogeneous.

Definition 2.1 *(Good perturbation).* Let F be an r-quasihomogeneous vector field of quasidegree $k-1$. A good perturbation X of F is a smooth vector field $X = F + P$, where $P = P(x_1, \ldots, x_{k-1}, z, \varepsilon)$ satisfies the following conditions:

- P is a smooth vector field of quasiorder greater than $k-1$,
- $P = \sum_{i=1}^{k-1} P_i \frac{\partial}{\partial x_i} + P_k \frac{\partial}{\partial z} + 0\frac{\partial}{\partial \varepsilon}$, with $P|_{\varepsilon=0} = 0$.

Notation By $\mathcal{P}_\delta$ we denote the space of r-quasihomogeneous polynomials (in $k+1$ variables) of quasidegree δ. By $\mathcal{H}_\gamma$ we denote the space of r-quasihomogeneous vector fields (in $\mathbb{R}^{k+1}$) of quasidegree γ and such that for all $U \in \mathcal{H}_\gamma$ we have $U = \sum_{i=1}^{k} U_k \frac{\partial}{\partial x_i} + 0\frac{\partial}{\partial x_{k+1}}$. The formal series expansion of a function f is denoted by $\hat{f}$.