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Algebra/Group theory

Quasi-hereditary property of double Burnside algebras

Propriété quasi-héréditaire des algèbres de Burnside doubles

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ABSTRACT

In this short note, we investigate some consequences of the *vanishing* of simple biset functors. As a corollary, if there is no non-trivial vanishing of simple biset functors (e.g., if the group G is commutative), then we show that $kB(G,G)$ is a *quasi-hereditary* algebra in characteristic zero. In general, this is not true without the non-vanishing condition, as over a field of characteristic zero, the double Burnside algebra of the alternating group of degree 5 has infinite global dimension.

RÉSUMÉ

Dans cette note, on s'intéresse à quelques conséquences du phénomène dit de *disparition* des foncteurs à bi-ensembles simples. On démontre que, dans le cas où il n'y a pas de disparitions non triviales de foncteurs simples (par exemple, si le groupe est commutatif), alors l'algèbre de Burnside double en caractéristique zéro est quasi-héréditaire. Sans l'hypothèse de non-disparitions triviales, ce résultat est en général faux. En effet, l'algèbre de Burnside double du groupe alterné de degré 5 en caractéristique zéro est de dimension globale infinie.

Notations. *Let k be a field. We denote by $\mathcal{C}_k$ the biset category. This is the category whose objects are finite groups and morphisms are given by the double Burnside module (see Definition 3.1.1 of [2]). For a finite group G, we denote by $\Sigma(G)$ the full subcategory of $\mathcal{C}_k$ consisting of the subquotients of G. If $\mathcal{D}$ is a k-linear subcategory of $\mathcal{C}_k$, we denote by $\mathcal{F}_{\mathcal{D},k}$ the category of k-linear functors from $\mathcal{D}$ to* k-Mod. *If L is a subquotient of K, we write $L \sqsubseteq K$ and if it is a proper subquotient, we write $L \sqsubset K$. If V and W are objects in the same Abelian category, we denote by $[V : W]$ the number of subquotients of V isomorphic to W.*

1. Evaluation of functors

Let us first recall some basic facts about the category of biset functors. Let $\mathcal{D}$ be an admissible subcategory of $\mathcal{C}_k$ in the sense of Definition 4.1.3 of [2]. The category $\mathcal{D}$ is a skeletally small k-linear category, so the category of biset functors is an Abelian category. The representable functors, also called Yoneda functors, are projective, so this category has *enough*

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projective. The simple functors are in bijection with the isomorphism classes of pairs (H, V), where H is an object of $\mathcal{D}$ and V is a $k\mathrm{Out}(H)$-simple module (see Theorem 4.3.10 of [2]).

A biset functor is called *finitely generated* if it is a quotient of a *finite* direct sum of representable functors. In particular, the simple biset functors and the representable functors are finitely generated. As in the case of modules over a ring, the choice axiom has for consequence the existence of a maximal subfunctor for finitely generated biset functors. If F is a biset functor, the intersection of all its maximal subfunctors is called the radical of F and denoted $\mathrm{Rad}(F)$.

If G is an object of $\mathcal{D}$, then there is an evaluation functor $ev_G : \mathcal{F}_{\mathcal{D},k} \to \mathrm{End}_{\mathcal{D}}(G)$-Mod sending a functor to its value at G. It is obviously an exact functor and it is well known that it sends a simple functor to 0 or to a simple $\mathrm{End}_{\mathcal{D}}(G)$-module. It turns out that the fact that a simple functor vanishes at G has some consequences for the functors having this simple as a quotient.

Proposition 1.1. *Let $F \in \mathcal{F}_{\mathcal{D},k}$ be a finitely generated functor and let $G \in Ob(\mathcal{D})$. Then*

1. $\mathrm{Rad}(F(G)) \subseteq [\mathrm{Rad}(F)](G)$.
2. *If none of the simple quotients of F vanishes at G, then* $\mathrm{Rad}(F(G)) = [\mathrm{Rad}(F)](G)$.

Proof. Let M be a maximal subfunctor of F. Then $M(G)$ is a maximal submodule of $F(G)$ if the simple quotient F/M does not vanish at G and $M(G) = F(G)$ otherwise. For the second part, if N is a maximal submodule of $F(G)$, let $\overline{N}$ be the subfunctor of F generated by N. There is a maximal subfunctor M of F such that $\overline{N} \subseteq M \subset F$. We have $\overline{N}(G) = N \subseteq M(G) \subset F(G)$. By maximality, $M(G) = N$. The result follows. □

Remark 1.2. In Section 9 of [3], the authors gave some conditions for the fact that the evaluation of the radical of the so-called standard functor is the radical of the evaluation. The elementary result of Proposition 1.1 gives new lights on this section. Indeed, Proposition 9.1 [3] gives a sufficient condition for the non-vanishing of the simple quotients of these standard functors.

Over a field, the category of finitely generated projective biset functors is Krull–Schmidt in the sense of [5] (Section 4), so every finitely generated biset functor has a projective cover.

Corollary 1.3. *Let $F \in \mathcal{F}_{\mathcal{D},k}$ be a finitely generated functor and let $G \in Ob(\mathcal{D})$. Then,*

1. *If F has a unique quotient S, and $S(G) \neq 0$, then $F(G)$ is an indecomposable* $\mathrm{End}_{\mathcal{D}}(G)$*-module.*
2. *If P is an indecomposable projective biset functor such that* $\mathrm{Top}(P)(G) \neq 0$*, then $P(G)$ is an indecomposable projective* $\mathrm{End}_{\mathcal{D}}(G)$*-module.*

2. Highest-weight structure of the biset functors category

Let us recall the famous theorem of Webb about the highest-weight structure of the category of biset functors.

Theorem 2.1. *(See Theorem 7.2 [6].) Let $\mathcal{D}$ be an admissible subcategory of the biset category. Let k be a field such that* $\mathrm{char}(k)$ *does not divide $|\mathrm{Out}(H)|$ for $H \in Ob(\mathcal{D})$. If $\mathcal{D}$ has a finite number of isomorphism classes of objects, then $\mathcal{F}_{\mathcal{D},k}$ is a highest-weight category.*

The set indexing the simple functors is the set, denoted by Λ, of isomorphism classes of pairs (H, V) where $H \in Ob(\mathcal{D})$ and V is a $k\mathrm{Out}(H)$-simple module. Let H and K be two objects of $\mathcal{D}$. Then

$$\bigoplus_{\substack{X \in \mathcal{D} \\ X \sqsubset H}} \mathrm{Hom}_{\mathcal{D}}(X, K)\mathrm{Hom}_{\mathcal{D}}(H, X),$$

can be viewed as a submodule of $\mathrm{Hom}_{\mathcal{D}}(H, K)$ via composition of morphisms. We denote by $I_{\mathcal{D}}(H, K)$ this submodule and by $\overleftarrow{\mathrm{Hom}}_{\mathcal{D}}(H, K)$ the quotient $\mathrm{Hom}_{\mathcal{D}}(H, K)/I_{\mathcal{D}}(H, K)$. This is a natural right $k\mathrm{Out}(H)$-module. If V is a $k\mathrm{Out}(H)$-module, we denote by $\Delta^{\mathcal{D}}_{H,V}$ the functor

$$\Delta^{\mathcal{D}}_{H,V} := K \mapsto \overleftarrow{\mathrm{Hom}}_{\mathcal{D}}(H, K) \otimes_{k\mathrm{Out}(H)} V.$$

When the context is clear, we simply denote by $\Delta_{H,V}$ this functor. If $(H, V) \in \Lambda$, then $\Delta_{H,V}$ is a standard object of $\mathcal{F}_{\mathcal{D},k}$. The set Λ is ordered by $(H, V) < (K, W)$ if $K \sqsubset H$, that is if K is a strict subquotient of H. So the highest-weight structure gives the fact that the projective indecomposable biset functors have a filtration by standard functors. This filtration has the following properties: