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Group theory

On the generation of discrete and topological Kac–Moody groups

Sur les générateurs des groupes de Kac–Moody topologiques et discrets

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ABSTRACT

This article shows that discrete or topological Kac–Moody groups defined over finite fields are in many cases 2-generated. We provide explicit bounds on the minimal number of generators for arbitrary Kac–Moody groups.

RÉSUMÉ

On montre que les groupes de Kac–Moody topologiques ou discrets définis sur des corps finis sont 2-engendrés dans de nombreux cas. On exhibe ensuite des bornes explicites sur le nombre minimal de générateurs pour un groupe de Kac–Moody arbitraire.

Version française abrégée

On considère des groupes de Kac–Moody sur des corps finis $\mathbb{F}_q$.

Théorème 0.1. *Soit $G = G(q)$ un groupe de Kac–Moody simplement connexe de rang m correspondant à une matrice de Cartan généralisée indécomposable (MCGI) A, défini sur un corps fini $\mathbb{F}_q$, $q = p^a$. Soit $\pi = \{\alpha_1, \dots, \alpha_m\}$ l'ensemble des racines simples de G et soit Δ le diagramme de Dynkin de G dont les sommets sont numérotées par $\alpha_1, \dots, \alpha_m$. Posons que, pour tout sous-ensemble σ de π non vide, $\Delta(\sigma)$ représente le sous-diagramme de Δ engendré par $\alpha_{i_1}, \dots, \alpha_{i_k} \in \pi$ où $\sigma = \{\alpha_{i_1}, \dots, \alpha_{i_k}\}$. Soit $d(G)$ le nombre minimal d'éléments de G nécessaires pour générer G. Alors lorsque q est suffisamment grand, on a :*

(i) lorsque $m = 2$, $d(G) \le 3$;
(ii) lorsque G est affine et que $m \ge 3$, $d(G) = 2$;
(iii) lorsque G est strictement hyperbolique (symétrisable) et $m \ge 3$, $d(G) = 2$;
(iv) lorsque G est hyperbolique (symétrisable), $d(G) = 2$ pour $m \ge 5$, et $d(G) \le 3$ si $m = 3$ ou $m = 4$ ($d(G) = 2$ dans au moins 34 des 72 cas) à part peut-être dans trois cas exceptionnels de rang 3 et pour lesquels Δ est de type (∞, ∞, ∞). Dans ces trois cas, $d(G) \le 4$;

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(v) *supposons que π peut être découpé en k sous-ensembles mutuellement disjoints π_i, $1 \le i \le k$, tels que $\pi_i = \{\alpha_{i_1}, \ldots, \alpha_{i_{l(i)}}\}$ avec $\alpha_{i_j} \in \pi$, $1 \le j \le l(i)$ (où $l(i) = |\pi_i|$) et que pour chaque $i \in \{1, \ldots, k-1\}$, on a $\Delta(\pi_i) = \bigsqcup_{j=1}^{s(i)} \Delta_{ij}$ où Δ_{ij} est un diagramme de Dynkin irréductible de type fini (ce qui signifie que $\Delta(\pi_i)$ peut être découpé en $s(i)$ diagrammes de Dynkin de type fini où $s(i) \in \mathbb{N}$ dépend de π_i). Alors :*
   (a) *si $\Delta(\pi_k) = \bigsqcup_{j=1}^{r} \Delta_{kj}$ où Δ_{kj} est un diagramme de Dynkin irréductible de type fini, alors $d(G) \le 2k$;*
   (b) *si $\Delta(\pi_k) = \bigsqcup_{j=1}^{r} \Delta_{kj}$ où Δ_{kj} est un diagramme de Dynkin irréductible de rang 2 de type infini, alors $d(G) \le 2k+2$, et, si q est assez grand, $d(G) \le 2k+1$.*

Exemple 1. Si Δ est un arbre enraciné fini de profondeur m, $d(G) \le 4$ lorsque $q \ge \sqrt{m}$.

Corollaire 0.2. *Soit G un groupe de Kac–Moody minimal défini sur un corps $\mathbb{F}_q$, avec $q = p^a$ et $p \ge \max_{i \ne j} |a_{ij}|$ (où $A = (a_{ij})$ est la MCGI de G). Soit $\overline{G}$ le groupe de Kac–Moody topologique correspondant à G. Alors les conclusions du Théorème 0.1 sont vraies si on remplace G par $\overline{G}$ et si $d(\overline{G})$ représente le nombre minimal de générateurs topologiques de $\overline{G}$.*

1. Introduction

It is a well-known result that every non-Abelian finite simple group can be generated by only two elements (cf. [2]). It is interesting to see whether this statement is true for other classes of simple groups. For example, non-affine Kac–Moody groups (over finite fields) are known to be simple [6]. How many generators do they require? In this article, we discuss the generation of Kac–Moody groups $G(q)$ defined over finite fields $\mathbb{F}_q$ and show that it is often the case that they too are 2-generated.

Kac–Moody groups over arbitrary fields were defined by J. Tits [16]. In [1], Abramenko and Muhlherr have shown that with some restrictions (if the groups are 2-spherical, with some mild bounds on the size of $\mathbb{F}_q$), Kac–Moody groups over $\mathbb{F}_q$ are finitely presented with the number of generators depending on q and the Lie rank of $G(q)$.[1] In [4], the author has shown that the family of affine Kac–Moody groups over $\mathbb{F}_q$ (of rank at least 3) possesses bounded presentations: there exists $C > 0$ such that if $G(q)$ is an affine Kac–Moody group of rank at least 3 corresponding to an indecomposable generalised Cartan matrix (IGCM) and $q \ge 4$, then $G(q)$ has a presentation with $d(G)$ generators and $r(G)$ relations satisfying $d(G) + r(G) \le C$. Related results for other Kac–Moody groups over finite fields were also proved in [4]. As a consequence, the number of generators of a 2-spherical Kac–Moody group is independent of q and depends on the type of Dynkin diagram of $G(q)$ rather than on the rank of G. We make use of this observation to provide bounds on the minimal number of generators of $G(q)$.

Theorem 1.1. *Let $G = G(q)$ be a simply connected Kac–Moody group of rank m corresponding to an IGCM A and defined over a finite field $\mathbb{F}_q$. Let $\pi = \{\alpha_1, \ldots, \alpha_m\}$ be the set of simple roots of G and Δ be the Dynkin diagram of G whose vertices are labelled by $\alpha_1, \ldots, \alpha_m$. Suppose further that for any non-empty subset σ of π, $\Delta(\sigma)$ denotes the subdiagram of Δ spanned by $\alpha_{i_1}, \ldots, \alpha_{i_k} \in \pi$ where $\sigma = \{\alpha_{i_1}, \ldots, \alpha_{i_k}\}$. Let $d(G)$ denote the minimal number of elements of G that are required to generate G. Then for q large enough there holds:*

(i) *if $m = 2$, then $d(G) \le 3$;*
(ii) *if G is affine with $m \ge 3$, then $d(G) = 2$;*
(iii) *if G is (symmetrizable) strictly hyperbolic and $m \ge 3$, then $d(G) = 2$;*
(iv) *if G is (symmetrizable) hyperbolic, then if $m \ge 5$, then $d(G) = 2$, and if $m = 3$ or 4, then $d(G) \le 3$ (with $d(G) = 2$ in at least 34 out of 72 cases) with the possible exception of three rank-3 diagrams with Δ of type (∞, ∞, ∞). In each one of those three cases, $d(G) \le 4$;*
(v) *suppose that we may subdivide π into k mutually disjoint subsets π_i, $1 \le i \le k$, such that each $\pi_i = \{\alpha_{i_1}, \ldots, \alpha_{i_{l(i)}}\}$ for some $\alpha_{i_j} \in \pi$, $1 \le j \le l(i)$ (with $l(i) = |\pi_i|$) and for each $i \in \{1, \ldots, k-1\}$, $\Delta(\pi_i) = \bigsqcup_{j=1}^{s(i)} \Delta_{ij}$ with Δ_{ij} an irreducible Dynkin diagram of finite type (i.e., $\Delta(\pi_i)$ can be partitioned into $s(i)$ disjoint Dynkin diagrams of finite type for some $s(i) \in \mathbb{N}$ depending on π_i). Then*
   (a) *if $\Delta(\pi_k) = \bigsqcup_{j=1}^{s(k)} \Delta_{kj}$ with Δ_{kj} an irreducible Dynkin diagram of finite type, then $d(G) \le 2k$;*
   (b) *if $\Delta(\pi_k) = \bigsqcup_{j=1}^{s(k)} \Delta_{kj}$ with Δ_{kj} an irreducible Dynkin diagram of rank 2 of infinite type, then $d(G) \le 2k+2$ (and if we increase q, $d(G) \le 2k+1$).*

The bound $d(G) = 2$ is optimal and was obtained in cases (*ii*), (*iii*) and part of (*iv*). Note that the bound $d(G) \le 2m$ follows from $(v)(a)$. Below are few examples of application of $(v)(a)$.

[1] An existence of finite generating set of $G(q)$ can be derived directly from the original presentation of $G(q)$.