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Differential geometry

Stability of holomorphically parallelizable manifolds

Stabilité des variétés holomorphiquement parallélisables

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A R T I C L E I N F O

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A B S T R A C T

We prove a stability theorem for families of holomorphically parallelizable manifolds in the category of Hermitian manifolds.

R É S U M É

Nous montrons un théorème de stabilité pour les familles de variétés holomorphiquement parallélisables, dans la catégorie des variétés hermitiennes.

1. Introduction

By a classical theorem by K. Kodaira and D.C. Spencer, [13, Theorem 15], small deformations of compact complex manifolds admitting a Kähler metric still admit Kähler metrics. This is a consequence of the harmonicity property of Kähler metrics and of Hodge theory on compact Kähler manifolds. On the other side, H. Hironaka provided in [12] an example of a complex-analytic family of compact complex manifolds being Kähler except for the central fibre, which is only Moĭšhezon. In other words, Kählerness is not a closed property under deformations. It is expected that limits of projective manifolds are Moĭšhezon, and limits of Kähler manifolds are in class $\mathcal{C}$ of Fujiki, see [9,16]. Note that $\partial\bar{\partial}$-Lemma is an invariant property under images of holomorphic birational maps, [8, Theorem 5.22]. In particular, compact complex manifolds being Moĭšhezon or belonging to class $\mathcal{C}$ of Fujiki satisfy the $\partial\bar{\partial}$-Lemma, [8, Corollary 5.23]. Hence, in view of the above conjectures, in [3,5], the behaviour of the $\partial\bar{\partial}$-Lemma property under deformation is investigated. In particular, [5, Corollary 2.7] provides another argument for proving the stability of $\partial\bar{\partial}$-Lemma under small deformations; see the references therein for different proofs, while an example showing that $\partial\bar{\partial}$-Lemma is not stable under limits is provided in [3, §4, Corollary 6.1]. Note in fact that

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the structures on the holomorphically parallelizable Nakamura manifold studied in [3, §4] are not in class $\mathcal{C}$ of Fujiki. In any case, nilmanifolds and solvmanifolds (that is, compact quotients of connected simply-connected nilpotent, respectively solvable, Lie groups by co-compact discrete subgroups) provide a possibly useful class of examples for investigating the above questions. In fact, Kählerness for nilmanifolds is characterized by $\partial\bar{\partial}$-Lemma, see [11, Theorem 1, Corollary], which is in turn characterized in terms of Bott–Chern cohomology, [5, Theorem B]. On the other hand, several results concerning the computation of Bott–Chern cohomology for nilmanifolds and solvmanifolds are known, see, e.g., [4,2] and the references therein. When restricting to the class of nilmanifolds, Kählerness is a closed properties under deformations. This follows by a theorem by A. Andreotti, H. Grauert, and W. Stoll in [1]. More precisely, they proved a stability result for complex-analytic families of complex tori.

In this short note, we prove a result similar to that by A. Andreotti, H. Grauert, and W. Stoll for the class of holomorphically parallelizable manifolds, in the category of compact Hermitian manifolds. As pointed out by the Referee, it remains an open question whether the result may be stated in the category of compact complex manifolds.

2. Main results

A compact complex manifold is called *holomorphically parallelizable* if its holomorphic tangent bundle is holomorphically trivial, see [19, page 771]. A structure theorem for holomorphically parallelizable manifolds was proven by H.-C. Wang. More precisely, holomorphically parallelizable manifolds have a complex Lie group as universal covering.

Theorem 2.1. *(See [19, Theorem 1].) Let X be a holomorphically parallelizable manifold. Then X is (biholomorphic to) a quotient G/D where G is a connected simply-connected complex Lie group and D is a discrete subgroup.*

Holomorphically parallelizable manifolds having a complex solvable Lie group as universal covering were studied by I. Nakamura in [14]. He initiated a classification of holomorphically parallelizable solvmanifolds up to complex dimension 5 in [14, §6], then completed by D. Guan in [10]. Moreover, by explicitly constructing the Kuranishi family of deformations of some holomorphically parallelizable solvmanifolds of complex dimension 3, in [14, §3], it was proved that being holomorphically parallelizable is not a stable property under small deformations of the complex structure, [14, page 86]. A detailed study of holomorphically parallelizable nilmanifolds, and of their Kuranishi space and stability was done by S. Rollenske in [17].

The structure theorem by H.-C. Wang allows us to generalize and simplify a stability result by A. Andreotti, H. Grauert, and W. Stoll, [1, Theorem 8]. In the proof below, the classical and well-known Montel theorem (also called generalized Vitali theorem) is used.

Theorem 2.2 *(Montel theorem). (See, e.g., [15, Proposition 6].) Let $\mathcal{F} = \{f\}$ be a family of holomorphic functions on an open set $\Omega \subseteq \mathbb{C}^n$ such that, for any compact set $K \subseteq \Omega$, there exists a positive constant M_K such that, for any $z \in K$, for any $f \in \mathcal{F}$, it holds $|f(z)| < M_K$. Then any sequence $\{f_n\}_n \subseteq \mathcal{F}$ contains a subsequence that converges uniformly on compact subsets of Ω.*

We can now state and prove the main result of this note.

Theorem 2.3. *Let $\{(X_t, g_t)\}_{t\in(-\varepsilon,1)}$ be a smooth family of compact Hermitian manifolds, with $\varepsilon > 0$ small enough. Suppose that X_t is holomorphically parallelizable for any $t \in (0,1)$, with a pointwise g_t-orthonormal co-frame $\left\{\varphi^j(t)\right\}_j$ of holomorphic 1-forms. Then X_0 is holomorphically parallelizable.*

Proof. First of all, by the Ehresmann theorem, for any $t \in (-\varepsilon, 1)$, we see $X_t = (X, J_t)$ where $\{J_t\}_{t\in(-\varepsilon,1)}$ is a family of complex structures on the differentiable manifold X varying smoothly in t.

By definition, the holomorphic tangent bundle $T^{1,0}X_t$ of X_t is holomorphically-trivial for any $t \in (0,1)$. Equivalently, the holomorphic co-tangent bundle $\left(T^{1,0}X_t\right)^*$ of X_t is holomorphically-trivial for any $t \in (0,1)$. Hence, by the assumptions, we choose $\left\{\varphi^1(t), \ldots, \varphi^n(t)\right\}$ global pointwise g_t-orthonormal co-frame of holomorphic 1-forms on X_t depending smoothly on t, where n denotes the complex dimension of X_t.

Denote by $(\cdot,\cdot\cdot)_t$ the induced L^2-Hermitian product on 1-forms, defined as $(\varphi,\psi)_t := \int_X \varphi \wedge *_{g_t}\bar{\psi}$, where $*_{g_t}$ denotes the Hodge-$*$-operator associated with g_t.

For any fixed $z_0 \in X_0$, consider a local holomorphic coordinate chart

$$\left(U \times (-\delta,\delta),\ \left(z^1 = x^1 + \mathrm{i}\,x^2, \ldots, z^n = x^{2n-1} + \mathrm{i}\,x^{2n}, t\right)\right)$$

centred at $(z_0, 0)$ on $\{X_t\}_{t\in(-\varepsilon,1)}$. Locally on $U \times (0,\delta)$, for $j \in \{1,\ldots,n\}$,

$$\varphi^j(z,t) \stackrel{\text{loc}}{=} \sum_{\alpha=1}^{n} \varphi^j_\alpha(z^1,\ldots,z^n,t)\,\mathrm{d}z^\alpha \qquad \text{in } U \times (0,\delta)\,,$$

where $\left\{\varphi^j_\alpha(z^1,\ldots,z^n,t)\right\}_\alpha$ are smooth in $(z^1,\ldots,z^n,t)$ and holomorphic in $(z^1,\ldots,z^n)$.