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Sharp estimates of integral functionals on classes of functions with small mean oscillation

Estimations précises de certaines fonctionnelles intégrales sur des classes de fonctions avec une petite oscillation moyenne

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ABSTRACT

We unify several Bellman function problems treated in [1,2,4–6,9–12,14–16,18–25]. For that purpose, we define a class of functions that have, in a sense, small mean oscillation (this class depends on two convex sets in $\mathbb{R}^2$). We show how the unit ball in the BMO space, or a Muckenhoupt class, or a Gehring class can be described in such a fashion. Finally, we consider a Bellman function problem on these classes, discuss its solution and related questions.

RÉSUMÉ

Nous unifions plusieurs problèmes concernant la fonction de Bellman traités dans [1,2,4–6,9–12,14–16,18–25]. Dans ce but, nous introduisons une classe de fonctions dont l'oscillation moyenne est petite dans un certain sens (cette classe depend de deux sous-ensembles convexes de $\mathbb{R}^2$). Nous démontrons que la boule unité de l'espace BMO, ou de la classe de Muckenhoupt, ou de la classe de Gehring, peut être décrite de cette façon. Finalement, nous considérons un problème de fonction de Bellman sur chacune de ces classes et discutons sa résolution ainsi que des questions voisines.

Since Slavin [12] and Vasyunin [18] proved the sharp form of the John–Nirenberg inequality (see [15]), there have been many papers where similar principles were used to prove sharp estimates of this kind. However, there is no theory or even a unifying approach; moreover, the class of problems to which the method can be applied has not been described yet.

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There is a portion of heuristics in the folklore that is each time applied to a new problem in a very similar manner. The first attempt to build a theory (at least for BMO) was made in [16], then the theory was developed in the paper [4] (see the short report [5] also). We would also like to draw the reader's attention to the preprint [6], which can be considered as a description of the theory for the BMO space in a sufficient generality. Problems of this kind were considered not only in BMO, but in Muckenhoupt classes, Gehring classes, etc. (see [1,2,11,13,19,20]). In this short note, we define a class of functions and an extremal problem on it that includes all the problems discussed above.

1. Setting

Let Ω_0 be a non-empty open strictly convex subset of $\mathbb{R}^2$ and let Ω_1 be an open strictly convex subset of Ω_0. We define the domain Ω as $cl(\Omega_0 \setminus \Omega_1)$ (the word "domain" comes from "domain of a function"; the symbol *cl* denotes the closure) and the class $\boldsymbol{A}_\Omega$ of summable $\mathbb{R}^2$-valued functions on an interval $I \subset \mathbb{R}$ as follows:

$$\boldsymbol{A}_\Omega = \big\{\varphi \in L^1(I,\mathbb{R}^2) \mid \varphi(I) \subset \partial\Omega_0 \quad \text{and} \quad \forall \text{ subinterval } J \subset I \quad \langle\varphi\rangle_J \notin \Omega_1\big\}. \tag{1}$$

Here $\langle\varphi\rangle_J = \frac{1}{|J|}\int_J \varphi(s)\,ds$ is the average of φ over J. In Section 2 we show how the unit ball in BMO as well as the "unit balls" in Muckenhoupt and Gehring classes can be represented in the form (1). Let $f\colon \partial\Omega_0 \to \mathbb{R}$ be a bounded from below Borel measurable locally bounded function. We are interested in sharp bounds for the expressions of the form $\langle f(\varphi)\rangle_I$, where $\varphi \in \boldsymbol{A}_\Omega$.

Again, in Section 2 we explain how the John–Nirenberg inequality or other inequalities of harmonic analysis can be rewritten as estimations of such an expression. The said estimates are delivered by the corresponding Bellman function

$$\boldsymbol{B}_{\Omega,f}(x) = \sup\big\{\langle f(\varphi)\rangle_I \,\big|\, \langle\varphi\rangle_I = x,\ \varphi \in \boldsymbol{A}_\Omega\big\}. \tag{2}$$

Problem 1.1. Given a domain Ω and a function f, calculate the function $\boldsymbol{B}_{\Omega,f}$.

The particular cases of this problem were treated in the papers [1,2,4–6,9–12,14–16,18–25] (see Section 2 for a detailed explanation). The main reason for Problem 1.1 to be solvable (and it has been heavily used in all the preceeding work) is that the function $\boldsymbol{B}$ enjoys good properties.

Definition 1.2. Let ω be a subset of $\mathbb{R}^d$. We call a function $G\colon w \to \mathbb{R}\cup\{+\infty\}$ locally concave on ω if for every segment $\ell \subset \omega$ the restriction $G\big|_\ell$ is concave.

Define the class of functions on Ω:

$$\Lambda_{\Omega,f} = \Big\{G\colon \Omega \to \mathbb{R}\cup\{+\infty\} \,\Big|\, G \text{ is locally concave on } \Omega,\ \ \forall x \in \partial\Omega_0 \quad G(x) \geqslant f(x)\Big\}. \tag{3}$$

The function $\mathfrak{B}_{\Omega,f}$ is given as follows:

$$\mathfrak{B}_{\Omega,f}(x) = \inf_{G \in \Lambda_{\Omega,f}} G(x). \tag{4}$$

Conjecture 1.3. $\boldsymbol{B}_{\Omega,f} = \mathfrak{B}_{\Omega,f}$.

In particular, the conjecture states that the Bellman function is locally concave (because the function $\mathfrak{B}_{\Omega,f}$ is).

Problem 1.4. Prove Conjecture 1.3 in adequate generality.

Though it may seem that one should solve Problem 1.4 before turning to Problem 1.1, it is not really the case. All the preceding papers used Conjecture 1.3 as an assumption that allowed the authors to guess $\boldsymbol{B}$, then to prove that this function was the Bellman function indeed, and only then verify Conjecture 1.3 for Ω and f chosen. However, to treat Problem 1.4 in itself, one has to invent a different approach, see Section 3.

2. Examples

From now on, we follow the agreement: if $g\colon \mathbb{R} \to \mathbb{R}^2$ is some fixed parameterization of $\partial\Omega_0$, then the function $f(g)\colon \mathbb{R} \to \mathbb{R}$ is denoted by $\tilde{f}$.

The BMO **space.** We consider the BMO space with the quadratic seminorm. Let ε be a positive number. Set $\Omega_0 = \{x \in \mathbb{R}^2 \mid x_1^2 < x_2\}$ and $\Omega_1 = \{x \in \mathbb{R}^2 \mid x_1^2 + \varepsilon^2 < x_2\}$. A function

$$\varphi = (\varphi_1, \varphi_2)\colon I \to \partial\Omega_0$$