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Mathematical analysis

Profile decomposition and phase control for circle-valued maps in one dimension

Décomposition en profils et contrôle des phases des applications unimodulaires en dimension un

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A B S T R A C T

When $1 < p < \infty$, maps f in $W^{1/p,p}((0,1);\mathbb{S}^1)$ have $W^{1/p,p}$ phases φ, but the $W^{1/p,p}$-seminorm of φ is not controlled by the one of f. Lack of control is illustrated by "the kink": $f = e^{\imath\varphi}$, where the phase φ moves quickly from 0 to 2π. A similar situation occurs for maps $f : \mathbb{S}^1 \to \mathbb{S}^1$, with Moebius maps playing the role of kinks. We prove that this is the only loss of control mechanism: each map $f : \mathbb{S}^1 \to \mathbb{S}^1$ satisfying $|f|^p_{W^{1/p,p}} \le M$ can be written as $f = e^{\imath\psi}\prod_{j=1}^{K}(M_{a_j})^{\pm 1}$, where M_{a_j} is a Moebius map vanishing at $a_j \in \mathbb{D}$, while the integer $K = K(f)$ and the phase ψ are controlled by M. In particular, we have $K \le c_p M$ for some c_p. When $p = 2$, we obtain the sharp value of c_2, which is $c_2 = 1/(4\pi^2)$. As an application, we obtain the existence of minimal maps of degree one in $W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)$ with $p \in (2-\varepsilon, 2)$.

R É S U M É

Si $1 < p < \infty$, les applications f appartenant à $W^{1/p,p}((0,1);\mathbb{S}^1)$ ont des phases φ dans $W^{1/p,p}$, mais la seminorme $W^{1/p,p}$ de φ n'est pas contrôlée par celle de f. L'absence de contrôle est illustrée par « le pli » : $f = e^{\imath\varphi}$, où la phase φ augmente rapidement de 0 à 2π. Pour des applications $f : \mathbb{S}^1 \to \mathbb{S}^1$, le même phénomène apparaît, avec les transformations de Moebius jouant le rôle des plis. Nous prouvons que cet exemple est essentiellement le seul : toute application $f : \mathbb{S}^1 \to \mathbb{S}^1$ telle que $|f|^p_{W^{1/p,p}} \le M$ s'écrit $f = e^{\imath\psi}\prod_{j=1}^{K}(M_{a_j})^{\pm 1}$, où M_{a_j} est une transformation de Moebius s'annulant en $a_j \in \mathbb{D}$, tandis que l'entier $K = K(f)$ et la phase ψ sont contrôlés par M. En particulier, nous avons $K \le c_p M$ pour une constante c_p. Pour $p = 2$, nous obtenons la valeur optimale de c_2, qui est $c_2 = 1/(4\pi^2)$.

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Comme application, nous obtenons l'existence d'une application minimale de degré un dans $W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)$ avec $p\in]2-\varepsilon, 2[$.

1. Introduction

Let $0<s<1$, $1\le p<\infty$ and let $f:(0,1)\to\mathbb{S}^1$ belong to the space $W^{s,p}$. Then f can be written as $f=e^{\imath\varphi}$, where $\varphi\in W^{s,p}$ [3]. Once the existence of φ is known, a natural question is whether we can control $|\varphi|_{W^{s,p}}$ in terms of $|f|_{W^{s,p}}$. For most of s, p, the answer is positive. The exceptional cases are provided precisely by the spaces $W^{1/p,p}((0,1);\mathbb{S}^1)$, with $1<p<\infty$ [3]. In these spaces, lack of control is established via the following explicit example. For $n\ge 1$, we define φ_n as follows:

$$\varphi_n(x):=\begin{cases}0, & \text{for } 0<x<1/2\\ 2\pi n(x-1/2), & \text{for } 1/2<x<1/2+1/n\\ 2\pi, & \text{for } 1/2+1/n<x<1\end{cases}.$$

Then $|\varphi_n|_{W^{1/p,p}}\to\infty$ (since $\varphi_n\to\varphi=2\pi\,\chi_{(1/2,1)}$ a.e., and φ does not belong to $W^{1/p,p}$). On the other hand, if we extend $u_n:=e^{\imath\varphi_n}$ with the value 1 outside $(0,1)$ and still denote the extension u_n then, by scaling,

$$|u_n|_{W^{1/p,p}((0,1))}\le |u_n|_{W^{1/p,p}(\mathbb{R})}=|u_1|_{W^{1/p,p}(\mathbb{R})}<\infty.$$

Thus $|u_n|_{W^{1/p,p}((0,1))}\lesssim 1$ and $|\varphi_n|_{W^{1/p,p}((0,1))}\to\infty$. Finally, we invoke the fact that $W^{1/p,p}$ phases are unique mod 2π [3].

If one considers instead maps $f:\mathbb{S}^1\to\mathbb{S}^1$, always in the critical case $f\in W^{1/p,p}$, $1<p<\infty$, then a new phenomenon occurs: f has a degree $\deg f$, and does not have a $W^{1/p,p}$ phase at all when $\deg f\ne 0$ [11, Remark 10]. However, even if $\deg f=0$ (and thus f has a $W^{1/p,p}$ phase φ), we have a loss-of-control phenomenon similar to the one on $(0,1)$. Indeed, let $M_a(z):=\dfrac{a-z}{1-\overline{a}z}$, $a\in\mathbb{D}$, $z\in\overline{\mathbb{D}}$, be a Moebius transform (that we identify with its restriction to $\mathbb{S}^1$, $M_a:\mathbb{S}^1\to\mathbb{S}^1$). Let $f_a(z):=\overline{z}M_a(z)$, so that f_a is smooth and $\deg f_a=0$. One may prove (see below) that $|M_a|_{W^{1/p,p}}=|\mathrm{Id}|_{W^{1/p,p}}$, and thus f_a is bounded in $W^{1/p,p}$. However, if $a\to\alpha=e^{\imath\xi}\in\mathbb{S}^1$, then the smooth phase φ_a of f_a converges a.e. to $\varphi(e^{\imath\theta}):=\begin{cases}\xi-\theta, & \text{if } \xi-\pi<\theta<\xi\\ 2\pi+\xi-\theta, & \text{if } \xi<\theta<\xi+\pi\end{cases}$, which does not belong to $W^{1/p,p}$. (Here, uniqueness of the phases and convergence hold mod 2π.) Thus φ_a is not bounded as $a\to\alpha\in\mathbb{S}^1$. On the other hand, the plot of φ_a shows that φ_a has a "kink shape", and thus we have here the analog of the example on $(0,1)$.

There are evidences that this loss of control mechanism is the only possible one. For example, the phase of the kink is not bounded in $W^{1/p,p}$, but clearly is in $W^{1,1}$ (same for f_a). Bourgain and Brézis [4] proved that for every $f\in W^{1/2,2}((0,1);\mathbb{S}^1)$, we may split $f=e^{\imath\psi}v$, with ψ and $v=e^{\imath\eta}$ satisfying

$$|\psi|_{W^{1/2,2}}\lesssim |f|_{W^{1/2,2}} \text{ and } |\eta|_{W^{1,1}}=|v|_{W^{1,1}}\lesssim |f|^2_{W^{1/2,2}}. \tag{1}$$

Intuitively, one should think at v as at "the kink part of f". The above result was extended by Nguyen [18] to $1<p<\infty$: for every $1<p<\infty$ and every $f\in W^{1/p,p}((0,1);\mathbb{S}^1)$, we may split $f=e^{\imath\psi}v$, with ψ and $v=e^{\imath\eta}$ satisfying

$$|\psi|_{W^{1/p,p}}\le C_p\,|f|_{W^{1/p,p}} \text{ and } |\eta|_{W^{1,1}}=|v|_{W^{1,1}}\le C_p\,|f|^p_{W^{1/p,p}}. \tag{2}$$

Here we present another result in this direction, written for simplicity on the unit circle.

Theorem 1. *Let $1<p<\infty$ and $M>0$. Then there exist constants c_p and $F(M)$ such that: every map $f\in W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)$ satisfying $|f|^p_{W^{1/p,p}}\le M$ can be written as $f=e^{\imath\psi}\prod_{j=1}^{K}(M_{a_j})^{\varepsilon_j}$, with $\varepsilon_j\in\{-1,1\}$,*

$$K\le c_p\,M, \tag{3}$$

and

$$|\psi|^p_{W^{1/p,p}}\le F(M). \tag{4}$$

When $p=2$, we may take $c_2=1/(4\pi^2)$, and this constant is optimal.

Corollary 1. *Let $1<p<\infty$ and let $f_n, f\in W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)$ be such that $f_n\rightharpoonup f$ in $W^{1/p,p}$. Then, up to a subsequence, there exist $K\in\mathbb{N}$, $\varepsilon_j\in\{-1,1\}$, $a_{j_n}\in\mathbb{D}$, $\alpha_j\in\mathbb{S}^1$, $j=1,\dots,K$, $\psi_n\in W^{1/p,p}(\mathbb{S}^1;\mathbb{R})$, and a constant C, such that:*