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Some characterizations of the quasi-sum production models with proportional marginal rate of substitution

Certaines caractérisations des modèles de production quasi-somme avec un taux marginal de substitution proportionnelle

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This paper is dedicated to Prof. Ieronim Mihăilă on the occasion of his 79th birthday

ABSTRACT

In this note we classify quasi-sum production functions with constant elasticity of production with respect to any factor of production and with proportional marginal rate of substitution.

RÉSUMÉ

Dans cette note, nous classons les fonctions de production quasi-somme avec élasticité constante de la production par rapport à un facteur de production et avec un taux marginal de substitution proportionnel.

1. Introduction

The notion of *production function* is a key concept in both macroeconomics and microeconomics, being used in the mathematical modeling of the relationship between the output of a firm, an industry, or an entire economy, and the inputs that have been used in obtaining it. Generally, production function is a twice differentiable mapping $f : \mathbb{R}^n_+ \to \mathbb{R}_+$, $f = f(x_1, \ldots, x_n)$, where f is the quantity of output, n is the number of the inputs and $x_1, \ldots, x_n$ are the factor inputs. A production function f is called *quasi-sum* [3,5] if there are strict monotone functions $G, h_1, \ldots, h_n$ with $G' > 0$ such that

$$f(x) = G(h_1(x_1) + \ldots + h_n(x_n)), \tag{1}$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n_+$. We note that these functions are of great interest because they appear as solutions to the general bisymmetry equation, being related to the problem of consistent aggregation [1].

Among the family of production functions, the most famous is the so-called Cobb–Douglas production function. A generalized Cobb–Douglas production function depending on n-inputs is given by

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$$f(x_1,\ldots,x_n)=A\cdot\prod_{i=1}^{n}x_i^{\alpha_i},\tag{2}$$

where $A,\alpha_1,\ldots,\alpha_n>0$. We recall that a production function of the form $f(x)=G(h(x_1,\ldots,x_n))$, where G is a strictly increasing function and h is a homogeneous function of any given degree p, is said to be a *homothetic* production function [7]. It is easy to see that a production function f can be identified with the graph of f, *i.e.* the nonparametric hypersurface of $\mathbb{E}^{n+1}$ defined by

$$L(x_1,\ldots,x_n)=(x_1,\ldots,x_n,f(x_1,\ldots,x_n))\tag{3}$$

and called the *production hypersurface* of f (see [9,11]). Motivated by some recent classification results concerning production hypersurfaces [2,5,7,8,12], in the present work we classify quasi-sum production functions with a proportional marginal rate of substitution and investigate the existence of such production models whose production hypersurfaces have null Gauss–Kronecker curvature or null mean curvature. We recall that, if f is a production function with n inputs $x_1,x_2,\ldots,x_n$, $n\geq 2$, the *elasticity of production* with respect to a certain factor of production x_i is defined as

$$E_{x_i}=\frac{x_i}{f}f_{x_i}\tag{4}$$

and the *marginal rate of technical substitution* of input x_j for input x_i is given by

$$\mathrm{MRS}_{ij}=\frac{f_{x_j}}{f_{x_i}},\tag{5}$$

where the subscripts denote partial derivatives of the function f with respect to the corresponding variables. A production function satisfies *the proportional marginal rate of substitution property* if

$$\mathrm{MRS}_{ij}=\frac{x_i}{x_j},\ \text{for all } 1\leq i\neq j\leq n.\tag{6}$$

In the last section of the paper we will prove the following theorem that generalizes the results from [10].

Theorem 1.1. *Let f be a quasi-sum production function given by (1). Then:*

i. *The elasticity of production is a constant k_i with respect to a certain factor of production x_i if and only if f reduces to*

$$f(x_1,\ldots,x_n)=A\cdot x_i^{k_i}\cdot\exp\left(D\sum_{j\neq i}h_j(x_j)\right),\tag{7}$$

where A and D are positive constants.

ii. *The elasticity of production is a constant k_i with respect to all factors of production x_i, $i=1,\ldots,n$, if and only if f reduces to the generalized Cobb–Douglas production function given by (2).*

iii. *The production function satisfies the proportional marginal rate of substitution property if and only if it reduces to the homothetic generalized Cobb–Douglas production function given by*

$$f(x_1,\ldots,x_n)=F\left(\prod_{i=1}^{n}x_i^{k}\right),\tag{8}$$

where k is a nonzero real number.

iv. *If the production function satisfies the proportional marginal rate of substitution property, then:*

iv_1. *The production hypersurface has vanishing Gauss–Kronecker curvature if and only if, up to a suitable translation, f reduces to the following generalized Cobb–Douglas production function with constant return to scale:*

$$f(x_1,\ldots,x_n)=A\cdot\prod_{i=1}^{n}x_i^{\frac{1}{n}}.\tag{9}$$

iv_2. *The production hypersurface cannot be minimal.*

iv_3. *The production hypersurface has vanishing sectional curvature if and only if, up to a suitable translation, f reduces to the following generalized Cobb–Douglas production function:*

$$f(x_1,\ldots,x_n)=A\cdot\prod_{i=1}^{n}\sqrt{x_i}.\tag{10}$$