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Probability theory

Conditionally Gaussian stochastic integrals

Intégrales stochastiques conditionnellement gaussiennes

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ABSTRACT

We derive conditional Gaussian type identities of the form

$$E\left[\exp\left(\mathrm{i}\int_0^T u_t\,\mathrm{d}B_t\right)\,\Bigg|\,\int_0^T |u_t|^2\mathrm{d}t\right]=\exp\left(-\frac{1}{2}\int_0^T |u_t|^2\mathrm{d}t\right),$$

for Brownian stochastic integrals, under conditions on the process $(u_t)_{t\in[0,T]}$ specified using the Malliavin calculus. This applies in particular to the quadratic Brownian integral $\int_0^t AB_s\,\mathrm{d}B_s$ under the matrix condition $A^\dagger A^2=0$, using a characterization of Yor [6].

RÉSUMÉ

Nous obtenons des identités gaussiennes conditionnelles de la forme

$$E\left[\exp\left(\mathrm{i}\int_0^T u_t\,\mathrm{d}B_t\right)\,\Bigg|\,\int_0^T |u_t|^2\mathrm{d}t\right]=\exp\left(-\frac{1}{2}\int_0^T |u_t|^2\mathrm{d}t\right),$$

pour les intégrales stochastiques browniennes, sous des conditions sur le processus $(u_t)_{t\in[0,T]}$ exprimées à l'aide du calcul de Malliavin. Ces résultats s'appliquent en particulier à l'intégrale brownienne quadratique $\int_0^t AB_s\,\mathrm{d}B_s$ sous la condition matricielle $A^\dagger A^2=0$, en utilisant une caractérisation de Yor [6].

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1. Introduction

Let $(B_t)_{t\in[0,T]}$ be a d-dimensional Brownian motion generating the filtration $(\mathcal{F}_t)_{t\in[0,T]}$. When A is a $d\times d$ skew-symmetric matrix, the identity

$$E\left[\exp\left(\mathrm{i}\int_0^T AB_s\,\mathrm{d}B_s\right)\Bigg|\,B_t\right]=E\left[\exp\left(-\frac{1}{2}\int_0^T|AB_s|^2\,\mathrm{d}s\right)\Bigg|\,B_t\right],\tag{1}$$

$0\le t\le T$, has been proved in Theorem 2.1 of [1], extending a formula of [7] for the computation of the characteristic function of Lévy's stochastic area in case $d=2$.

This approach is connected to a result of Yor [6] stating that when $A^\dagger A^2=0$, the filtration $(\mathcal{F}_t^k)_{t\in[0,T]}$ of $t\mapsto\int_0^t AB_s\,\mathrm{d}B_s$ is generated by k independent Brownian motions, where k is the number of distinct eigenvalues of $A^\dagger A$.

In this Note, we derive conditional versions of the identity (1) for the stochastic integral $\int_0^T u_t\,\mathrm{d}B_t$ of an $(\mathcal{F}_t)$-adapted process $(u_t)_{t\in[0,T]}$ in Theorem 1, under conditions formulated in terms of the Malliavin calculus, using the cumulant–moment formula of [3,4]. In particular we provide conditions for $\int_0^T u_t\,\mathrm{d}B_t$ to be Gaussian $\mathcal{N}\left(0,\int_0^T|u_t|^2\,\mathrm{d}t\right)$-distributed given $\int_0^T|u_t|^2\,\mathrm{d}t$, cf. Theorem 2. This holds for example when $(u_t)_{t\in[0,T]}=(AB_t)_{t\in[0,T]}$ under Yor's condition $A^\dagger A^2=0$, cf. Corollary 3. We also consider a weakening of this condition to $A^\dagger A^2$ skew-symmetric, provided that $A^\dagger A$ is proportional to a projection, cf. Corollary 6.

2. Conditional characteristic functions

Let D denote the Malliavin gradient with domain $\mathbb{D}_{2,1}$ on the d-dimensional Wiener space, cf. § 1.2 of [2] for definitions. Taking $H=L^2([0,T];\mathbb{R}^d)$ for some $T>0$ and u in the domain $\mathbb{D}_{k,1}(H)$ of D in $L^k(\Omega;H)$, we let

$$(Du)^k u_t:=\int_0^T\cdots\int_0^T(D_{t_k}u_t)^\dagger(D_{t_{k-1}}u_{t_k})^\dagger\cdots(D_{t_1}u_{t_2})^\dagger u_{t_1}\,\mathrm{d}t_1\cdots\mathrm{d}t_k,\quad t\in[0,T],\quad k\ge1.$$

Theorem 1. *Let $u\in\bigcap_{k\ge1}\mathbb{D}_{k,1}(H)$ be an $(\mathcal{F}_t)$-adapted process such that*

$$\langle u_t,(Du)^k u_t\rangle_{\mathbb{R}^d}=0,\qquad t\in[0,T],\quad k\ge1.$$

We have

$$E\left[\exp\left(\mathrm{i}\int_0^T u_t\,\mathrm{d}B_t\right)\Bigg|\,(|u_t|)_{t\in[0,T]}\right]=\exp\left(-\frac{1}{2}\int_0^T|u_t|^2\,\mathrm{d}t\right),\tag{2}$$

provided that $\frac{1}{2}\int_0^T|u_t|^2\,\mathrm{d}t$ is exponentially integrable.

Proof. For any $F\in\mathbb{D}_{2,1}$ and $k\ge1$, let

$$\Gamma_k^u F:=\mathbb{1}_{\{k\ge2\}}F\int_0^T\langle u_t,(Du)^{k-2}u_t\rangle_{\mathbb{R}^d}\,\mathrm{d}t+\int_0^T\langle D_tF,(Du)^{k-1}u_t\rangle_{\mathbb{R}^d}\,\mathrm{d}t.$$

Recall that for any $u\in\mathbb{D}_{2,1}(H)$ such that $\Gamma_{l_1}^u\cdots\Gamma_{l_k}^u\mathbb{1}$ has finite expectation for all $l_1+\cdots+l_k\le n$, $k=1,\dots,n$, by Theorem 1 of [3] or Proposition 4.3 of [4] we have

$$E\left[F\left(\int_0^T u_t\,\mathrm{d}B_t\right)^n\right]=n!\sum_{a=1}^n\sum_{\substack{l_1+\cdots+l_a=n\\ l_1\ge1,\dots,l_a\ge1}}\frac{E\left[\Gamma_{l_1}^u\cdots\Gamma_{l_a}^u F\right]}{l_1(l_1+l_2)\cdots(l_1+\cdots+l_a)},\tag{3}$$

for $F\in\mathbb{D}_{2,1}$. Next, for any $f\in\mathcal{C}_b^1(\mathbb{R})$ and $k\ge1$ we have