C. R. Acad. Sci. Paris, Ser. I 353 (2015) 887–892

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Algebra/Lie algebras

Generalized Joseph's decompositions ☆

Décompositions de Joseph généralisées

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ARTICLE INFO

Article history:
Received 23 April 2015
Accepted after revision 9 July 2015
Available online 6 August 2015

Presented by the Editorial Board

ABSTRACT

We generalize the decomposition of $U_q(\mathfrak{g})$ introduced by A. Joseph in [5] and link it, for $\mathfrak{g}$ semisimple, to the celebrated computation of central elements due to V. Drinfeld [2]. In that case, we construct a natural basis in the center of $U_q(\mathfrak{g})$ whose elements behave as Schur polynomials and thus explicitly identify the center with the ring of symmetric functions.

RÉSUMÉ

Nous généralisons la décomposition de $U_q(\mathfrak{g})$ introduite par A. Joseph [5] et la relions, pour $\mathfrak{g}$ semi-simple, au calcul bien connu d'éléments centraux dû à V. Drinfeld [2]. Dans ce cas, nous construisons une base naturelle dans le centre de $U_q(\mathfrak{g})$, dont les éléments se conduisent comme des polynômes de Schur, et nous identifions donc explicitement le centre avec l'anneau de fonctions symétriques.

1. Introduction and main results

1.1. Let H be an associative algebra with unity over a field $\Bbbk$ and let $\mathscr{C}$ be a full abelian subcategory closed under submodules of the category $H-\mathrm{Mod}$ of left H-modules. Suppose that we have a "finite duality" functor $^\star:\mathscr{C}\to \mathrm{Mod}-H$ with $V^\star\subseteq V^*=\mathrm{Hom}_{\Bbbk}(V,\Bbbk)$ (with equality if and only if V is finite dimensional) with its natural right H-module structure, such that the restriction of the evaluation pairing $\langle\cdot,\cdot\rangle_V : V\otimes V^*\to\Bbbk$ to $V\otimes V^\star$ is non-degenerate for all objects V in $\mathscr{C}$ (see Section 2.1 for details). Following [4], we define $\beta_V : V\otimes_{D(V)} V^\star\to H^*$ where $D(V)=\mathrm{End}_H V^\star=(\mathrm{End}_H V)^{\mathrm{op}}$ by

$$\beta_V(v\otimes f)(h)=\langle h\triangleright v, f\rangle_V=\langle v, f\triangleleft h\rangle_V,\qquad v\in V,\ f\in V^\star,\ h\in H,$$

where $\triangleright$ (respectively, $\triangleleft$) denotes the left (respectively, right) H-action. It is easy to see that β_V is well-defined. Set $H^*_V=\mathrm{Im}\beta_V$. Recall that $V\otimes V^\star$ and H^* are naturally H-bimodules. The following is essentially proved in [4, §3.1] and [3, Corollary 1.16].

☆ The authors are partially supported by the NSF grant DMS-1403527 (A. B.) and by the Simons Foundation collaboration grant no. 245735 (J. G.).
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http://dx.doi.org/10.1016/j.crma.2015.07.002

Proposition 1.1.

(a) *For all $V \in \mathscr{C}$, β_V is a homomorphism of H-bimodules and H_V^* depends only on the isomorphism class of V. Moreover, if $V, V' \in \mathscr{C}$ are simple and $H_V^* = H_{V'}^*$, then $V \cong V'$;*
(b) *$H_{V \oplus V'}^* = H_V^* + H_{V'}^*$ for all $V, V' \in \mathscr{C}$. In particular, $H_{V^{\oplus n}}^* = H_V^*$ for all $n \in \mathbb{N}$.*
(c) *If $V \otimes_{D(V)} V^\star$ is simple as an H-bimodule then β_V is injective.*
(d) *If V is simple finite dimensional, then $V \otimes_{D(V)} V^\star$ is simple as an H-bimodule and hence β_V is injective.*

It is natural to call H_V^* a *generalized Peter–Weyl component*. Denote $H_{\mathscr{C}}^* = \sum_{[V] \in \operatorname{Iso}\mathscr{C}} H_V^*$ and $\underline{H}_{\mathscr{C}}^* = \bigoplus_{[V] \in \operatorname{Iso}^\circ \mathscr{C}} H_V^*$, where $\operatorname{Iso}\mathscr{C}$ (respectively, $\operatorname{Iso}^\circ \mathscr{C}$) is the set of isomorphism classes of objects (respectively, simple objects) in $\mathscr{C}$. By definition, there is a natural homomorphism of H-bimodules $\underline{H}_{\mathscr{C}}^* \to H_{\mathscr{C}}^*$. Clearly, under the assumptions of Proposition 1.1(c), it is injective. Note that $H_{\mathscr{C}}^* = \sum_{[V] \in A} H_V^*$ for any subset A of $\operatorname{Iso}\mathscr{C}$, which generates it as an additive monoid. The following refinement of [4, Theorem 3.10] establishes the generalized Peter–Weyl decomposition.

Theorem 1.2. *Suppose that all objects in $\mathscr{C}$ have finite length. Then*

(a) *if $H_{\mathscr{C}}^* = \underline{H}_{\mathscr{C}}^*$ then $\mathscr{C}$ is semisimple;*
(b) *if $\mathscr{C}$ is semisimple and $V \otimes_{D(V)} V^\star$ is simple for every $V \in \mathscr{C}$ simple then $H_{\mathscr{C}}^* = \underline{H}_{\mathscr{C}}^*$.*

1.2. Henceforth we denote by $\mathscr{C}^{\text{fin}}$ the full subcategory of $\mathscr{C}$ consisting of all finite-dimensional objects. Clearly $V \otimes V^\star$, $V \in \mathscr{C}^{\text{fin}}$, is a unital algebra with unity 1_V; set $z_V := \beta_V(1_V) \in H_V^*$. For example, if $H = \Bbbk G$ for a finite group G, then for any finite-dimensional H-module V, we have $z_V(g) = tr_V(g)$, $g \in G$, where tr_V denotes the trace of a linear endomorphism of V.

Given an H-bimodule B, define the subspace B^H of H-invariants in B by $B^H = \{b \in B : h \triangleright b = b \triangleleft h, \forall h \in H\}$ (B^H is sometimes referred to as the center of B). Clearly, $z_V \in (H_V^*)^H$, $z_V(1_H) = \dim_\Bbbk V \neq 0$ and $(H_V^*)^H = \Bbbk z_V$ if $\operatorname{End}_H V = \Bbbk \operatorname{id}_V$. Set $\mathcal{Z}_{\mathscr{C}} = \sum_{[V] \in \operatorname{Iso}\mathscr{C}} \mathbb{Z} z_V$. Given $V \in \mathscr{C}$, denote $|V|$ its image in the Grothendieck group $K_0(\mathscr{C})$ of $\mathscr{C}$. The following result contrasts sharply with Proposition 1.1 and Theorem 1.2 for non-semisimple $\mathscr{C}$.

Theorem 1.3. *Suppose that $\mathscr{C} = \mathscr{C}^{\text{fin}}$. Then the map $K_0(\mathscr{C}) \to \mathcal{Z}_{\mathscr{C}}$ given by $|V| \mapsto z_V$, $[V] \in \operatorname{Iso}\mathscr{C}$ is an isomorphism of abelian groups.*

1.3. To introduce a multiplication on $\mathcal{Z}_{\mathscr{C}} \subset (H_{\mathscr{C}}^*)^H \subset H_{\mathscr{C}}^*$, we assume henceforth that $H = (H, m, \Delta, \varepsilon)$ is a bialgebra and that $\mathscr{C}$ is a tensor subcategory of $H - \text{Mod}$. Note that H^* is an algebra in a natural way. It is easy to see (Lemma 2.4) that $(H^*)^H$ is a subalgebra of H^*. We also assume that there is a natural isomorphism $(V \otimes V')^\star \cong V'^\star \otimes V^\star$ in $\text{mod} - H$ for all $V, V' \in \mathscr{C}$.

Theorem 1.4.

(a) *$H_V^* \cdot H_{V'}^* = H_{V \otimes V'}^*$ for all $V, V' \in \mathscr{C}$. In particular, $H_{\mathscr{C}}^*$ is a subalgebra of H^*;*
(b) *$z_V \cdot z_{V'} = z_{V \otimes V'}$ for all $V, V' \in \mathscr{C}^{\text{fin}}$. In particular, if $\mathscr{C} = \mathscr{C}^{\text{fin}}$ then $\mathcal{Z}_{\mathscr{C}}$ is a subring of $(H_{\mathscr{C}}^*)^H$ and the map $K_0(\mathscr{C}) \to \mathcal{Z}_{\mathscr{C}}$ from Theorem 1.3 is an isomorphism of rings.*

Thus, it is natural to regard $\mathcal{Z}_{\mathscr{C}}$ as the character ring of $\mathscr{C}$.

1.4. It turns out that we can transfer the above structures from $H_{\mathscr{C}}^*$ to H if $H = (H, m, \Delta, \varepsilon, S)$ is a Hopf algebra. For an H-bimodule B, define left H-actions ad and $\diamond$ on B via $(\operatorname{ad} h)(b) = h_{(1)} \triangleright b \triangleleft S(h_{(2)})$ and $h \diamond b = S^2(h_{(2)}) \triangleright b \triangleleft S(h_{(1)})$, $h \in H$, $b \in B$, where $\Delta(b) = b_{(1)} \otimes b_{(2)}$ in Sweedler's notation.

Fix a categorical completion $H \widehat{\otimes} H$ of $H \otimes H$ such that $(f \otimes 1)(H \widehat{\otimes} H) \subset H$ for all $f \in H_{\mathscr{C}}^*$. Equivalently, $\Phi_P : H_{\mathscr{C}}^* \to H$, $f \mapsto (f \otimes 1)(P)$ is a well-defined linear map. Denote $\mathscr{A}(H)$ the set of all $P \in H \widehat{\otimes} H$ such that $P \cdot (S^2 \otimes 1)(\Delta(h)) = \Delta(h) \cdot P$ for all $h \in H$. Clearly, $\mathscr{A}(H)$ is a subalgebra of $H \widehat{\otimes} H$. Elements of $\mathscr{A}(H)$ are analogous to M-matrices (see, e.g., [12]). For $V \in \mathscr{C}^{\text{fin}}$, set $c_V = c_{V,P} := \Phi_P(z_V) \in \Phi_P((H_{\mathscr{C}}^*)^H)$. Let $Z(H)$ be the center of H.

Theorem 1.5. *Let $P \in \mathscr{A}(H)$. Then $\Phi_P : H_{\mathscr{C}}^* \to H$ is a homomorphism of left H-modules, where H acts on $H_{\mathscr{C}}^*$ and H via $\diamond$ and* ad, *respectively. Moreover, $\Phi_P((H_{\mathscr{C}}^*)^H) \subset Z(H)$ and the assignment $|V| \mapsto c_V$, $[V] \in \operatorname{Iso}\mathscr{C}^{\text{fin}}$ defines a homomorphism of abelian groups $\operatorname{ch}_{\mathscr{C}} : K_0(\mathscr{C}^{\text{fin}}) \to Z(H)$.*

Surprisingly, Φ_P is often close to be an algebra homomorphism. To make this more precise, we generalize the notion of an algebra homomorphism as follows. Let A, B be $\Bbbk$-algebras and let $\mathscr{F}$ be a collection of subspaces in A. We say that a $\Bbbk$-linear map $\Phi : A \to B$ is an *$\mathscr{F}$-homomorphism* if $\Phi(U) \cdot \Phi(U') \subset \Phi(U \cdot U')$ for all $U, U' \in \mathscr{F}$. We say that $\mathscr{F}$ is