C. R. Acad. Sci. Paris, Ser. I 353 (2015) 953–958

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Probability theory

An improvement of the mixing rates in a counter-example to the weak invariance principle

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ARTICLE INFO

Article history:
Received 9 February 2015
Accepted after revision 27 July 2015
Available online 1 October 2015

Presented by Jean-François Le Gall

ABSTRACT

In [1], the authors gave an example of absolutely regular strictly stationary process that satisfies the central limit theorem, but not the weak invariance principle. For each $q < /1/2$, the process can be constructed with mixing rates of order N^{-q}. The goal of this note is to show that actually the same construction can give mixing rates of order N^{-q} for a given $q < 1$.

RÉSUMÉ

Dans [1], les auteurs ont fourni un exemple de processus strictement stationnaire β-mélangeant vérifiant le théorème limite central, mais pas le principe d'invariance faible. Pour tout $q < 1/2$, le processus peut être construit avec des taux de mélange de l'ordre de N^{-q}. L'objectif de cette note est de montrer que la même construction peut fournir des taux de mélange de l'ordre de N^{-q} pour un $q < 1$ donné.

1. Notations and main result

We recall some notations in order to make this note more self-contained. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. If $T : \Omega \to \Omega$ is one-to-one, bi-measurable and measure preserving (in sense that $\mu(T^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{F}$), then the sequence $\left(f \circ T^k\right)_{k \in \mathbb{Z}}$ is strictly stationary for any measurable $f : \Omega \to \mathbb{R}$. Conversely, each strictly stationary sequence can be represented in this way.

For a zero mean square integrable $f : \Omega \to \mathbb{R}$, we define $S_n(f) := \sum_{j=0}^{n-1} f \circ T^j$, $\sigma_n^2(f) := \mathbb{E}(S_n(f)^2)$ and $S_n^*(f, t) := S_{\lfloor nt \rfloor}(f) + (nt - \lfloor nt \rfloor) f \circ T^{\lfloor nt \rfloor}$, where $\lfloor x \rfloor$ is the greatest integer, which is less than or equal to x.

Define the β-mixing coefficients by

$$\beta(\mathcal{A}, \mathcal{B}) := \frac{1}{2} \sup \sum_{i=1}^{I} \sum_{j=1}^{J} \left| \mu(A_i \cap B_j) - \mu(A_i)\mu(B_j) \right|, \tag{1}$$

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http://dx.doi.org/10.1016/j.crma.2015.07.013
1631-073X/

where the supremum is taken over the finite partitions $\{A_i, 1 \leqslant i \leqslant I\}$ and $\left\{B_j, 1 \leqslant j \leqslant J\right\}$ of Ω of elements of $\mathcal{A}$ (respectively of $\mathcal{B}$). They were introduced by Volkonskii and Rozanov [4].

For a strictly stationary sequence $(X_k)_{k\in\mathbb{Z}}$ and $n \geqslant 0$, we define $\beta_X(n) = \beta(n) = \beta(\mathcal{F}_{-\infty}^{0}, \mathcal{F}_{n}^{\infty})$, where $\mathcal{F}_u^v$ is the σ-algebra generated by X_k with $u \leqslant k \leqslant v$ (if $u = -\infty$ or $v = \infty$, the corresponding inequality is strict).

Theorem 1. *Let $\delta > 0$. There exists a strictly stationary real valued process $Y = (Y_k)_{k\geqslant 0} = \left(f \circ T^k\right)_{k\geqslant 0}$ satisfying the following conditions:*

a) *the central limit theorem with normalization $\sqrt{n}$ takes place;*
b) *the weak invariance principle with normalization $\sqrt{n}$ does not hold;*
c) *$\sigma_N(f)^2 \asymp N$;*
d) *for some positive C and each integer N, $\beta_Y(N) \leqslant C \cdot N^{-1+\delta}$;*
e) *$Y_0 \in \mathbb{L}^p$ for any $p > 0$.*

We refer the reader to Remark 2 of [1] for a comparison with existing results about the weak invariance principle for strictly stationary mixing sequences.

2. Proof

We recall the construction given in [1]. Let us consider an increasing sequence of positive integers $(n_k)_{k\geqslant 1}$ such that

$$n_1 \geqslant 2 \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{n_k} < \infty, \tag{2}$$

and for each integer $k \geqslant 1$, let A_k^-, A_k^+ be disjoint measurable sets such that $\mu(A_k^-) = 1/(2n_k^2) = \mu(A_k^+)$.

Let the random variables e_k be defined by

$$e_k(\omega) := \begin{cases} 1 & \text{if } \omega \in A_k^+, \\ -1 & \text{if } \omega \in A_k^-, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

We can choose the dynamical system $(\Omega, \mathcal{F}, \mu, T)$ and the sets A_k^+, A_k^- in such a way that the family $(e_k \circ T^i)_{k\geqslant 1, i\in\mathbb{Z}}$ is independent. We define $A_k := A_k^+ \cup A_k^-$ and

$$h_k := \sum_{i=0}^{n_k-1} U^{-i} e_k - U^{-n_k} \sum_{i=0}^{n_k-1} U^{-i} e_k, \quad h := \sum_{k=1}^{+\infty} h_k. \tag{4}$$

Let $i(N)$ denote the unique integer such that $n_{i(N)} \leqslant N < n_{i(N)+1}$.

We shall show the following intermediate result.

Proposition 1. *Assume that the sequence $(n_k)_{k\geqslant 1}$ satisfies (2) and the following condition:*

$$\text{there exists } \eta > 0 \text{ such that for each } k, \quad n_{k+1} \geqslant n_k^{1+\eta}. \tag{5}$$

Then:

a') *$n^{-1/2} S_n(h) \to 0$ in probability;*
b') *the process $(N^{-1/2} S_N^*(h, \cdot))_{N\geqslant 1}$ is not tight in $C[0, 1]$;*
c') *$\sigma_N(h)^2 \lesssim N$;*
d') *for some positive C, $N \cdot \beta_Y(N) \leqslant C n_{i(N)+1}/n_{i(N)}$;*
e') *$h \in \mathbb{L}^p$ for any $p > 0$.*

Let us consider a bounded mean-zero function m of unit variance such that the sequence $(m \circ T^i)_{i\geqslant 0}$ is independent and independent of the sequence $(h \circ T^i)_{i\geqslant 0}$. We define $f := m + h$.

Corollary 2. *Assume the sequence $(n_k)_{k\geqslant 1}$ satisfies (5). Then the sequence $(f \circ T^i)_{i\geqslant 0}$ satisfies a), b), c), d') and e).*